GENERALIZED DEMAZURE MODULES AND FUSION PRODUCTS

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ABSTRACT. Let $\mathfrak{g}$ be a finite-dimensional complex simple Lie algebra with highest root θ and let $\mathfrak{g}[t]$ be the corresponding current algebra. In this paper, we consider the $\mathfrak{g}[t]$-stable Demazure modules associated to integrable highest weight representations of the affine Lie algebra $\widehat{\mathfrak{g}}$. We prove that the fusion product of Demazure modules of a given level with a single Demazure module of a different level and with highest weight a multiple of θ is a generalized Demazure module, and also give defining relations. This also shows that the fusion product of such Demazure modules is independent of the chosen parameters. As a consequence we obtain generators and relations for certain types of generalized Demazure modules. We also establish a connection with the modules defined by Chari and Venkatesh.

1. Introduction

Let $\mathfrak{g}$ be a finite-dimensional complex simple Lie algebra and $\widehat{\mathfrak{g}}$ the corresponding affine Lie algebra. In this paper, we are interested in Demazure modules associated to integrable highest weight representations of $\widehat{\mathfrak{g}}$. These modules, which are actually modules for a Borel subalgebra of $\widehat{\mathfrak{g}}$, are indexed by a dominant integral affine weight and an element of the affine Weyl group. We are mainly interested in the Demazure modules which are preserved by a maximal parabolic subalgebra containing the Borel. The maximal parabolic subalgebra of our interest is the current algebra $\mathfrak{g}[t]$, which is the algebra of polynomial maps $\mathbb{C} \to \mathfrak{g}$ with the obvious point-wise bracket. Equivalently, it is the complex vector space $\mathfrak{g} \otimes \mathbb{C}[t]$ with Lie bracket being the $\mathbb{C}[t]$-bilinear extension of the Lie bracket on $\mathfrak{g}$. The degree grading on $\mathbb{C}[t]$ gives a natural $\mathbb{Z}$-grading on $\mathfrak{g}[t]$ and makes it a graded Lie algebra. The $\mathfrak{g}[t]$-stable Demazure modules are known to be indexed by pairs (ℓ, λ), where ℓ is the level of the integrable representation of $\widehat{\mathfrak{g}}$ and λ is a dominant integral weight of $\mathfrak{g}$. We denote the corresponding module by $D(\ell, \lambda)$. These are in fact finite-dimensional graded $\mathfrak{g}[t]$-modules.

A powerful tool to study the category of finite-dimensional graded $\mathfrak{g}[t]$-modules is the fusion product, which was introduced by Feigin and Loktev in [5]. Although the fusion product is by definition dependent on a choice of parameters, it is widely expected that it will turn out to be independent of the choices, and in several cases this has been proved (see [3, 4, 7, 10, 13, 14, 16]). It is proved in [3] that the fusion product of Demazure modules of a given level is again a Demazure module of the same level. In [14], for $\mathfrak{g}$ simply laced, it is proved that the fusion product of Demazure modules of different level with highest weight a multiple of a fundamental weight is a generalized Demazure module, and used this to solve the $X = M$ conjecture. The generalized Demazure modules are indexed by p dominant integral affine weights and p affine Weyl group elements, where $p \geq 1$. Their defining relations are not known except when $p = 1$, where they

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are actually the Demazure modules. But in special cases the character of generalized Demazure modules is known in terms of the Demazure operators [11, 15].

In this paper, we investigate further the fusion product of different level Demazure modules. Let θ be the highest root of $\mathfrak{g}$. We consider the fusion product of Demazure modules of a given level with a single Demazure module of a different level and with highest weight a multiple of θ. We prove that this fusion product as a $\mathfrak{g}[t]$-module is isomorphic to a generalized Demazure module, and give the defining relations. More precisely, given positive integers k, ℓ, m such that $\ell \geq m \geq k$, and a sequence of dominant integral weights $\lambda_1, \ldots, \lambda_p$ of $\mathfrak{g}$, we prove that the fusion product

$$D(\ell,\, \ell\lambda_1) * \cdots * D(\ell,\, \ell\lambda_p) * D(m,\, k\theta), \tag{1.1}$$

of Demazure modules is a generalized Demazure module, and also give the defining relations (see Theorem 3.3). This also proves that the fusion product (1.1) is independent of the chosen parameters.

Our main results (Theorems 3.2 and 3.3) enable us to obtain short exact sequences of fusion products and generalized Demazure modules (see Corollary 3.4), and a surjective morphism between two fusion products (see Corollary 3.5). As a consequence of Corollary 3.5, we get the following result which may be viewed as a generalization of the Schur positivity [1] (see Corollary 3.6): Given two partitions $(\ell_1 \geq \ell_2 \geq \cdots \geq \ell_p \geq 0)$ and $(m_1 \geq m_2 \geq \cdots \geq m_p \geq 0)$ of a positive integer, there exists a surjective morphism of $\mathfrak{g}$-modules

$$D(\ell_1,\, \ell_1\theta) \otimes D(\ell_2,\, \ell_2\theta) \otimes \cdots \otimes D(\ell_p,\, \ell_p\theta) \twoheadrightarrow D(m_1,\, m_1\theta) \otimes D(m_2,\, m_2\theta) \otimes \cdots \otimes D(m_p,\, m_p\theta),$$

if $\ell_i + \cdots + \ell_p \geq m_i + \cdots + m_p$ holds for each $1 \leq i \leq p$.

In [4], Chari and Venkatesh have introduced a large family of indecomposable graded $\mathfrak{g}[t]$-modules which includes the Demazure modules. In §5, we prove that certain types of generalized Demazure modules also belong to their family (see Corollary 5.3).

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2. Preliminaries

Throughout the paper, $\mathbb{Z}$ denotes the set of integers, $\mathbb{N}$ the set of positive integers, $\mathbb{Z}_{\geq 0}$ the set of non-negative integers, $\mathbb{C}$ the field of complex numbers, $\mathbb{C}[t]$ the polynomial ring in an indeterminate t and $\mathbb{C}[t, t^{-1}]$ the ring of Laurent polynomials.

2.1. Given a complex Lie algebra $\mathfrak{a}$, let $\mathbf{U}(\mathfrak{a})$ be its universal enveloping algebra. The current algebra $\mathfrak{a}[t]$ associated to $\mathfrak{a}$ is defined as $\mathfrak{a} \otimes \mathbb{C}[t]$, with the Lie bracket

$$[x \otimes t^r, y \otimes t^s] = [x, y] \otimes t^{r+s} \quad \forall\ x, y \in \mathfrak{a},\ r, s \in \mathbb{Z}_{\geq 0}.$$

The degree grading on $\mathbb{C}[t]$ gives a natural $\mathbb{Z}_{\geq 0}$-grading on $\mathbf{U}(\mathfrak{a}[t])$: the element $(a_1 \otimes t^{s_1}) \cdots (a_k \otimes t^{s_k})$, for $a_i \in \mathfrak{a}, s_i \in \mathbb{Z}_{\geq 0}$, has grade $s_1 + \cdots + s_k$. A graded $\mathfrak{a}[t]$-module is a $\mathbb{Z}$-graded vector space $V = \bigoplus_{r \in \mathbb{Z}} V[r]$ such that

$$(\mathfrak{a} \otimes t^s)V[r] \subset V[r+s], \quad \forall\ r \in \mathbb{Z},\ s \in \mathbb{Z}_{\geq 0}.$$

Let $\mathrm{ev}_0 : \mathfrak{a}[t] \to \mathfrak{a}$ be the morphism of Lie algebras given by setting $t = 0$. The pull back of any $\mathfrak{a}$-module V by ev_0 defines a graded $\mathfrak{a}[t]$-module structure on V, and we denote this module by $\mathrm{ev}_0 V$. We define the morphism of graded $\mathfrak{a}[t]$-modules as a degree zero morphism of $\mathfrak{a}[t]$-modules. For $r \in \mathbb{Z}$ and a graded $\mathfrak{a}[t]$-module V, we let $\tau_r V$ be the r-th graded shift of V.

2.2. Let $\mathfrak{g}$ be a finite-dimensional simple Lie algebra over $\mathbb{C}$ of rank n. Fix a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ and a Borel subalgebra $\mathfrak{b}$ of $\mathfrak{g}$ containing $\mathfrak{h}$. Let $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+$ be the triangular decomposition of $\mathfrak{g}$ with $\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}^+$. Let R (resp. R^+) be the set of roots (resp. positive roots) of $\mathfrak{g}$. Let $\theta \in R^+$ be the highest root of $\mathfrak{g}$. Let $(.|.)$ be a non-degenerate, symmetric, invariant bilinear form on $\mathfrak{h}^*$ normalized so that the square length of a long root is two. It is easy to see from the abstract theory of root systems that $(\theta|\alpha) \in \{0, 1\}$, for all $\alpha \in R^+ \setminus \{\theta\}$. For $\alpha \in R$, let $\alpha^\vee \in \mathfrak{h}$ denote the corresponding co-root, $\mathfrak{g}_\alpha$ the corresponding root space of $\mathfrak{g}$, and we fix non-zero elements $x_\alpha^\pm \in \mathfrak{g}_{\pm\alpha}$ such that $[x_\alpha^+, x_\alpha^-] = \alpha^\vee$. For a root α, we set $d_\alpha = \frac{2}{(\alpha|\alpha)}$, then we have

$$d_\alpha = \begin{cases} 1 & \text{if } \alpha \text{ is long,} \\ 2 & \text{if } \alpha \text{ is short and } \mathfrak{g} \text{ is of type } B_n, C_n \text{ or } F_4, \\ 3 & \text{if } \alpha \text{ is short and } \mathfrak{g} \text{ is of type } G_2. \end{cases}$$

Set $I = \{1, 2, \ldots, n\}$. Let α_i and $\varpi_i, i \in I$, be simple roots and fundamental weights respectively. For $\alpha = \sum_i n_i \alpha_i \in R$, we define the *height* of α by $\mathrm{ht}\, \alpha = \sum_i n_i$. For $i \in I$, we write $x_i^\pm$ for $x_{\alpha_i}^\pm$ and d_i for d_{α_i}. The weight lattice P (resp. P^+) is the $\mathbb{Z}$-span (resp. $\mathbb{Z}_{\geq 0}$-span) of $\{\varpi_i : i \in I\}$. The root lattice Q (resp. Q^+) is the $\mathbb{Z}$-span (resp. $\mathbb{Z}_{\geq 0}$-span) of $\{\alpha_i : i \in I\}$. The co-weight lattice $L = \sum_{i \in I} \mathbb{Z} d_i \varpi_i$ is a sub lattice of P and the co-root lattice $M = \sum_{i \in I} \mathbb{Z} d_i \alpha_i$ is a sub lattice of Q. The subsets $L^+$and M^+ are defined in the obvious way. It is easy to see that for $\lambda \in L^+$ and $\alpha \in R^+$, we have $(\lambda|\alpha) \in \mathbb{Z}_{\geq 0}$. Let W be the Weyl group of $\mathfrak{g}$. For $\alpha \in R^+$, we denote by $r_\alpha \in W$ the reflection associated with α, and set $r_i = r_{\alpha_i} (i \in I)$. We denote by w_0 the longest element in W.

For $\lambda \in P^+$, let $V(\lambda)$ be the corresponding finite-dimensional irreducible $\mathfrak{g}$-module generated by an element v_λ with the following defining relations:

$$x_i^+ v_\lambda = 0, \quad \alpha_i^\vee v_\lambda = \langle \lambda, \alpha_i^\vee \rangle v_\lambda, \quad (x_i^-)^{\langle \lambda, \alpha_i^\vee \rangle + 1} v_\lambda = 0, \quad \forall\, i \in I.$$

2.3. Let $\widehat{\mathfrak{g}}$ be the affine Lie algebra defined by

$$\widehat{\mathfrak{g}} = \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}c \oplus \mathbb{C}d,$$

where c is central and the other Lie brackets are given by

$$[x \otimes t^r, y \otimes t^s] = [x, y] \otimes t^{r+s} + r\delta_{r,-s}(x|y)c,$$

$$[d, x \otimes t^r] = r(x \otimes t^r),$$

for all $x, y \in \mathfrak{g}$ and integers r, s. The Lie subalgebras $\widehat{\mathfrak{h}}$ and $\widehat{\mathfrak{b}}$ of $\widehat{\mathfrak{g}}$ are defined as follows:

$$\widehat{\mathfrak{h}} = \mathfrak{h} \oplus \mathbb{C}c \oplus \mathbb{C}d, \qquad \widehat{\mathfrak{b}} = \mathfrak{g} \otimes t\mathbb{C}[t] \oplus \mathfrak{b} \oplus \mathbb{C}c \oplus \mathbb{C}d.$$

We regard $\mathfrak{h}^*$ as a subspace of $\widehat{\mathfrak{h}}^*$ by setting $\langle \lambda, c\rangle = \langle \lambda, d\rangle = 0$ for $\lambda \in \mathfrak{h}^*$. For $\xi \in \widehat{\mathfrak{h}}^*$, let $\xi|_{\mathfrak{h}}$ be the element of $\mathfrak{h}^*$ obtained by restricting ξ to $\mathfrak{h}$. Let $\delta, \Lambda_0 \in \widehat{\mathfrak{h}}^*$ be given by

$$\langle \delta, \mathfrak{h} + \mathbb{C}c\rangle = 0,\ \langle \delta, d\rangle = 1, \quad \langle \Lambda_0, \mathfrak{h} + \mathbb{C}d\rangle = 0,\ \langle \Lambda_0, c\rangle = 1.$$

Extend the non-degenerate form on $\mathfrak{h}^*$ to a non-degenerate symmetric bilinear form on $\widehat{\mathfrak{h}}^*$ by setting,

$$(\mathfrak{h}^*|\mathbb{C}\delta + \mathbb{C}\Lambda_0) = (\delta|\delta) = (\Lambda_0|\Lambda_0) = 0 \text{ and } (\delta|\Lambda_0) = 1.$$

Set $\widehat{I} = I \cup \{0\}$. For $i \in I$, denote $\Lambda_i = \varpi_i + \langle \varpi_i, \theta^\vee\rangle\Lambda_0 \in \widehat{\mathfrak{h}}^*$. Let $\widehat{P}^+ = \sum_{i\in\widehat{I}} \mathbb{Z}_{\geq 0}(\Lambda_i + \mathbb{Z}\delta)$ be the set of dominant integral affine weights, and $\widehat{P}$ is defined similarly. The affine root lattice $\widehat{Q}$ is the $\mathbb{Z}$-span of the simple roots $\alpha_i, i \in \widehat{I}$, of $\widehat{\mathfrak{g}}$, and $\widehat{Q}^+$ is defined in the obvious way. Let $\widehat{R}_{re} = \{\alpha + r\delta : \alpha \in R,\ r \in \mathbb{Z}\}$ be the set of real roots, $\widehat{R}_{im} = \{r\delta : r \in \mathbb{Z}\setminus\{0\}\}$ the set of imaginary roots and $\widehat{R} = \widehat{R}_{re} \cup \widehat{R}_{im}$ the set of roots of $\widehat{\mathfrak{g}}$. For each real root $\alpha + r\delta$, we have the Lie subalgebra of $\widehat{\mathfrak{g}}$ generated by $\{x_\alpha^+ \otimes t^r, x_\alpha^- \otimes t^{-r}\}$ which is isomorphic to $\mathfrak{sl}_2$.

Let $\widehat{W}$ be the affine Weyl group with simple reflections $r_i (i \in \widehat{I})$. We regard W naturally as a subgroup of $\widehat{W}$. Given $\alpha \in \mathfrak{h}^*$, we define $t_\alpha \in GL(\widehat{\mathfrak{h}}^*)$ by

$$t_\alpha(\lambda) = \lambda + (\lambda|\delta)\,\alpha - (\lambda|\alpha)\,\delta - \frac{1}{2}\,(\lambda|\delta)\,(\alpha|\alpha)\,\delta \quad \text{for } \lambda \in \widehat{\mathfrak{h}}^*.$$

The translation subgroup T_M of $\widehat{W}$ is defined by $T_M = \{t_\alpha \in GL(\widehat{\mathfrak{h}}^*) : \alpha \in M\}$ and we have

$$\widehat{W} = W \ltimes T_M.$$

The extended affine Weyl group $\widetilde{W}$ is the semi-direct product

$$\widetilde{W} = W \ltimes T_L,$$

where $T_L = \{t_\beta \in GL(\widehat{\mathfrak{h}}^*) : \beta \in L\}$. We also have $\widetilde{W} = \widehat{W} \rtimes \Sigma$, where Σ is the subgroup of diagram automorphisms of $\widehat{\mathfrak{g}}$. Given $w \in \widehat{W}$, let $\ell(w)$ be the length of a reduced expression of w. The length function ℓ is extended to $\widetilde{W}$ by setting $\ell(w\sigma) = \ell(w)$ for $w \in \widehat{W}$ and $\sigma \in \Sigma$.

The following lemma is proved in [3], and will be required later.

Lemma 2.1. *[3, Proposition 2.8] Given $\lambda, \mu \in P^+$ and $w \in W$, we have*

$$\ell(t_{-\lambda}t_{-\mu}w) = \ell(t_{-\lambda}) + \ell(t_{-\mu}w).$$

For any group G, let $\mathbb{Z}[G]$ be the integral group ring of G with basis $e^g, g \in G$. Let I_δ be the ideal of $\mathbb{Z}[\widehat{P}]$ obtained by setting $e^\delta = 1$. For a finite-dimensional semisimple $\mathfrak{h}$-module V, we define $\mathfrak{h}$-character $\mathrm{ch}_{\mathfrak{h}} V$ by

$$\mathrm{ch}_{\mathfrak{h}} V = \sum_{\mu\in\mathfrak{h}^*} \dim V_\mu\, e^\mu \in \mathbb{Z}[\mathfrak{h}^*],$$

where $V_\mu = \{v \in V : hv = \langle \mu, h\rangle\, \forall\, h \in \mathfrak{h}\}$. For a finite-dimensional semisimple $\widehat{\mathfrak{h}}$-module V, the $\widehat{\mathfrak{h}}$-character $\mathrm{ch}_{\widehat{\mathfrak{h}}} V \in \mathbb{Z}[\widehat{\mathfrak{h}}^*]$ is defined in the similar way.

2.4. Let $V(\Lambda)$ be the integrable highest weight $\widehat{\mathfrak{g}}$-module corresponding to a dominant integral affine weight Λ. Let $\xi_1, \dots, \xi_p$ be a sequence of elements of $\widehat{W}(\widehat{P}^+)$. For $1 \le j \le p$, let Λ^j be the element of $\widehat{P}^+$ such that $\xi_j \in \widehat{W}\Lambda^j$. Define a $\widehat{\mathfrak{b}}$-submodule $D(\xi_1, \dots, \xi_p)$ of $V(\Lambda^1) \otimes \cdots \otimes V(\Lambda^p)$ by

$$D(\xi_1, \dots, \xi_p) = \mathbf{U}(\widehat{\mathfrak{b}})\big(V(\Lambda^1)_{\xi_1} \otimes \cdots \otimes V(\Lambda^p)_{\xi_p}\big).$$

We call the $\widehat{\mathfrak{b}}$-module $D(\xi_1, \dots, \xi_p)$ as a generalized Demazure module [11,15]. When $p = 1$, $D(\xi_1)$ is called as a Demazure module. We observe that $D(\xi_1, \dots, \xi_p) \subset D(\xi_1) \otimes \cdots \otimes D(\xi_p)$. Note that $D(\xi_1, \dots, \xi_p)$ is $\mathfrak{g}$-stable if $\langle \xi_j, \alpha_i^\vee \rangle \le 0, \ \forall\, i \in I, \ 1 \le j \le p$.

2.5. We now recall from [4, §3.5] the definition of the finite-dimensional graded $\mathfrak{g}[t]$-modules $D(\ell, \lambda)$, $(\ell, \lambda) \in \mathbb{N} \times P^+$. For $\alpha \in R^+$ with $\langle \lambda, \alpha^\vee \rangle > 0$, let $s_\alpha, m_\alpha \in \mathbb{N}$ be the unique positive integers such that

$$\langle \lambda, \alpha^\vee \rangle = (s_\alpha - 1) d_\alpha \ell + m_\alpha, \quad 0 < m_\alpha \le d_\alpha \ell.$$

If $\langle \lambda, \alpha^\vee \rangle = 0$ we set $s_\alpha = 1$ and $m_\alpha = 0$. The module $D(\ell, \lambda)$ is the cyclic $\mathfrak{g}[t]$-module generated by an element $w_{\ell,\lambda}$ with the following defining relations:

$$(x_i^+ \otimes t^s)\, w_{\ell,\lambda} = 0, \quad (\alpha_i^\vee \otimes t^s)\, w_{\ell,\lambda} = \langle \lambda, \alpha_i^\vee \rangle \delta_{s,0} w_{\ell,\lambda}, \quad (x_i^- \otimes 1)^{\langle \lambda, \alpha_i^\vee \rangle + 1}\, w_{\ell,\lambda} = 0, \quad \forall\ s \ge 0, i \in I,$$
$$(x_\alpha^- \otimes t^{s_\alpha})\, w_{\ell,\lambda} = 0, \quad \forall\, \alpha \in R^+,$$
$$(x_\alpha^- \otimes t^{s_\alpha - 1})^{m_\alpha + 1}\, w_{\ell,\lambda} = 0, \text{ if } m_\alpha < d_\alpha \ell, \quad \forall\, \alpha \in R^+.$$

The following relations also hold in the module $D(\ell, \lambda)$

$$(x_\alpha^- \otimes t^{s_\alpha - 1})^{m_\alpha + 1}\, w_{\ell,\lambda} = 0, \text{ if } m_\alpha = d_\alpha \ell, \quad \forall\, \alpha \in R^+.$$

We declare the grade of $w_{\ell,\lambda}$ to be zero. Since the defining relations of $D(\ell, \lambda)$ are graded, it follows that $D(\ell, \lambda)$ is a graded $\mathfrak{g}[t]$-module.

The following result, which gives the connection with Demazure modules, is a combination of results [4, Theorem 2], [7, Corollary 1], and [13, Proposition 3.6].

Proposition 2.2. *[4, 7, 13] Given $(\ell, \lambda) \in \mathbb{N} \times P^+$ and $m \in \mathbb{Z}$, let $w \in \widehat{W}, \sigma \in \Sigma$, and $\Lambda \in \widehat{P}^+$ such that*

$$w\sigma\Lambda \equiv w_0\lambda + \ell\Lambda_0 + m\delta.$$

Then we have the following isomorphism of $\mathfrak{g}[t]$-modules,

$$\tau_m D(\ell, \lambda) \cong D(w\sigma\Lambda).$$

Under this isomorphism the generator $w_{\ell,\lambda}$ of $\tau_m D(\ell, \lambda)$ maps to a non-zero element of the weight space of $D(w\sigma\Lambda)$ of weight $w_0 w\sigma\Lambda$.

2.6. We recall the notion of the fusion product of finite-dimensional cyclic graded $\mathfrak{g}[t]$-modules given in [5].

Let V be a cyclic $\mathfrak{g}[t]$-module generated by v. We define a filtration F^rV, $r \in \mathbb{Z}_{\geq 0}$ on V by

$$F^rV = \sum_{0\leq s\leq r} \mathbf{U}(\mathfrak{g}[t])[s]\, v.$$

Set $F^{-1}V = \{0\}$. The associated graded space $\operatorname{gr} V = \bigoplus_{r\geq 0} F^rV/F^{r-1}V$ naturally becomes a cyclic $\mathfrak{g}[t]$-module generated by the image of v in $\operatorname{gr} V$.

Given a $\mathfrak{g}[t]$-module V and a complex number z, we define an another $\mathfrak{g}[t]$-module action on V as follows:

$$(x \otimes t^s)\, v = (x \otimes (t+z)^s)\, v, \quad x \in \mathfrak{g},\ v \in V,\ s \in \mathbb{Z}_{\geq 0}.$$

Denote this new module by V^z. For $1 \leq i \leq m$, let V_i be a finite-dimensional cyclic graded $\mathfrak{g}[t]$-module generated by v_i. Let $z_1, \dots, z_m$ be distinct complex numbers. We denote $\mathbf{V} = {V_1}^{z_1} \otimes \cdots \otimes {V_m}^{z_m}$, the tensor product of the corresponding $\mathfrak{g}[t]$-modules. It is known (see [5, Proposition 1.4]) that $\mathbf{V}$ is a cyclic $\mathfrak{g}[t]$-module generated by $v_1 \otimes \cdots \otimes v_m$. The associated graded space $\operatorname{gr}\mathbf{V}$ is called the fusion product of $V_1, \dots, V_m$ w.r.t. the parameters $z_1, \dots, z_m$, and is denoted by ${V_1}^{z_1} * \cdots * {V_m}^{z_m}$. We denote the image of $v_1 \otimes \cdots \otimes v_m$ in $\operatorname{gr}\mathbf{V}$ by $v_1 * \cdots * v_m$. For ease of notation we shall often write $V_1 * \cdots * V_m$ for ${V_1}^{z_1} * \cdots * {V_m}^{z_m}$. We note that $V_1 * \cdots * V_m \cong_{\mathfrak{g}} V_1 \otimes \cdots \otimes V_m$.

3. The main results

We begin this section by introducing a class of finite-dimensional graded cyclic $\mathfrak{g}[t]$-modules and then state our main results.

Given $k, \ell, m \in \mathbb{Z}_{\geq 0}$ such that $k \leq m$ and $\lambda \in L^+$. We define $\mathbf{V}^{\ell,\,\ell\lambda}_{m,\,k\theta}$ to be the cyclic $\mathfrak{g}[t]$-module generated by an element $v = v^{\ell,\,\ell\lambda}_{m,\,k\theta}$ with the following defining relations:

$$(x_i^+ \otimes t^s)\, v = 0, \quad (\alpha_i^\vee \otimes t^s)\, v = \delta_{s,0}\langle \ell\lambda + k\theta,\, \alpha_i^\vee\rangle\, v, \quad \left(x_i^- \otimes 1\right)^{\langle \ell\lambda+k\theta,\, \alpha_i^\vee\rangle+1} v = 0, \quad \forall\ s \geq 0, i \in I, \tag{3.1}$$

$$\left(x_\alpha^- \otimes t^{(\lambda+\theta|\alpha)}\right) v = 0, \quad \forall\ \alpha \in R^+, \tag{3.2}$$

$$\left(x_\alpha^- \otimes t^{(\lambda|\alpha)}\right)^{\langle (k-i)\theta,\, \alpha^\vee\rangle+1} \left(x_\theta^- \otimes t^{(\lambda|\theta)+1}\right)^i v = 0, \quad \forall\ \alpha \in R^+,\ 0 \leq i \leq k, \tag{3.3}$$

$$\left(x_\theta^- \otimes t^{(\lambda|\theta)+1}\right)^{2k-m+1} v = 0, \quad \text{if } m \leq 2k, \tag{3.4}$$

$$\left(x_\theta^- \otimes t^{(\lambda|\theta)+1}\right) v = 0, \quad \text{if } m \geq 2k. \tag{3.5}$$

The relations (3.1) guarantee that the module $\mathbf{V}^{\ell,\,\ell\lambda}_{m,\,k\theta}$ is finite-dimensional (cf. [2]). In particular this gives,

$$\left(x_\alpha^- \otimes 1\right)^{\langle \ell\lambda+k\theta,\, \alpha^\vee\rangle+1} v = 0, \quad \forall\ \alpha \in R^+. \tag{3.6}$$

We declare the grade of v to be zero. Since the defining relations of $\mathbf{V}^{\ell,\,\ell\lambda}_{m,\,k\theta}$ are graded, it follows that $\mathbf{V}^{\ell,\,\ell\lambda}_{m,\,k\theta}$ is a graded $\mathfrak{g}[t]$-module.

Remark 3.1. *We prove in §5 that the relations in* (3.3)*, when $\alpha \in R^+ \setminus \{\theta\}$ and $1 \leq i \leq k$, are redundant (see Lemma 5.6).*

We observe that

$$\mathbf{V}^{\ell,\,\ell\lambda}_{m,\,k\theta} \cong_{\mathfrak{g}[t]} \mathbf{V}^{\ell,\,\ell\lambda}_{2k,\,k\theta}, \quad \forall\ m \geq 2k, \tag{3.7}$$

and

$$\mathbf{V}^{\ell,\,\ell\lambda}_{0,\,0\theta} \cong_{\mathfrak{g}[t]} D(\ell,\,\ell\lambda). \tag{3.8}$$

The following theorems are the main results of this paper.

Theorem 3.2. *Let $k, \ell, m \in \mathbb{N}$ and $\lambda \in L^+$.*

(1) If $\ell \geq 2k$, then there exists a short exact sequence of $\mathfrak{g}[t]$-modules,

$$0 \to \tau_{(\lambda|\theta)+1}\mathbf{V}^{\ell,\,\ell\lambda}_{k-1,\,(k-1)\theta} \xrightarrow{\phi_1^-} \mathbf{V}^{\ell,\,\ell\lambda}_{k,\,k\theta} \xrightarrow{\phi_k^+} \mathbf{V}^{\ell,\,\ell\lambda}_{\ell,\,k\theta} \to 0.$$

(2) If $\ell \geq 2k$ and $k < m \leq 2k$, then there exists a short exact sequence of $\mathfrak{g}[t]$-modules,

$$0 \to \tau_{(2k-m+1)((\lambda|\theta)+1)}\mathbf{V}^{\ell,\,\ell\lambda}_{m-k-1,\,(m-k-1)\theta} \xrightarrow{\psi^-} \mathbf{V}^{\ell,\,\ell\lambda}_{k,\,k\theta} \xrightarrow{\psi^+} \mathbf{V}^{\ell,\,\ell\lambda}_{m,\,k\theta} \to 0.$$

(3) If $\ell \leq 2k$ and $k \leq m < \ell$, then there exists a short exact sequence of $\mathfrak{g}[t]$-modules,

$$0 \to \tau_{(2k-\ell+1)((\lambda|\theta)+1)}\mathbf{V}^{\ell,\,\ell\lambda}_{\ell+m-2k-1,\,(\ell-k-1)\theta} \xrightarrow{\phi_2^-} \mathbf{V}^{\ell,\,\ell\lambda}_{m,\,k\theta} \xrightarrow{\phi_m^+} \mathbf{V}^{\ell,\,\ell\lambda}_{\ell,\,k\theta} \to 0.$$

Theorem 3.3. *Let $k, \ell, m \in \mathbb{N}$ be such that $\ell \geq m \geq k$. Let $\lambda_1, \ldots, \lambda_p$ be a sequence of elements of L^+, and denote $\lambda = \lambda_1 + \cdots + \lambda_p$. Then we have the following isomorphisms of $\mathfrak{g}[t]$-modules:*

$$\begin{aligned} &D(\ell,\,\ell\lambda_1) * \cdots * D(\ell,\,\ell\lambda_p) * D(m,\,k\theta) \\ &\cong \mathbf{V}^{\ell,\,\ell\lambda}_{m,\,k\theta} \cong \begin{cases} D\big(t_{w_0\lambda}(\ell-m)\Lambda_0,\, t_{w_0(\lambda+\theta)}(m\Lambda_0 + (m-k)\theta)\big), & m \leq 2k, \\ D\big(t_{w_0\lambda}(\ell-m)\Lambda_0,\, t_{w_0\lambda}w_0(m\Lambda_0 + k\theta)\big), & m \geq 2k. \end{cases} \end{aligned}$$

Theorems 3.2 and 3.3 are proved in §4.

The following corollary is immediate from Theorems 3.2 and 3.3.

Corollary 3.4. *Let $k, \ell, m \in \mathbb{N}$ and $\lambda_1, \ldots, \lambda_p \in L^+$. Denote $\lambda = \lambda_1 + \cdots + \lambda_p$.*

(1) If $\ell \geq 2k$, then there exist short exact sequences of $\mathfrak{g}[t]$-modules,

$$\begin{aligned} 0 &\to \tau_{(\lambda|\theta)+1}\big(D(\ell,\,\ell\lambda_1) * \cdots * D(\ell,\,\ell\lambda_p) * D(k-1,\,(k-1)\theta)\big) \\ &\to D(\ell,\,\ell\lambda_1) * \cdots * D(\ell,\,\ell\lambda_p) * D(k,\,k\theta) \\ &\to D(\ell,\,\ell\lambda_1) * \cdots * D(\ell,\,\ell\lambda_p) * D(\ell,\,k\theta) \to 0 \end{aligned}$$

and

$$\begin{aligned} 0 &\to \tau_{(\lambda|\theta)+1}\, D\big(t_{w_0\lambda}(\ell-k+1)\Lambda_0,\, t_{w_0(\lambda+\theta)}(k-1)\Lambda_0\big) \\ &\to D\big(t_{w_0\lambda}(\ell-k)\Lambda_0,\, t_{w_0(\lambda+\theta)}k\Lambda_0\big) \\ &\to D\big(t_{w_0(\lambda+\theta)}(\ell\Lambda_0 + (\ell-k)\theta)\big) \to 0. \end{aligned}$$

(2) If $\ell \geq 2k$ and $k < m \leq 2k$, then there exist short exact sequences of $\mathfrak{g}[t]$-modules,

$$\begin{aligned} 0 &\to \tau_{(2k-m+1)((\lambda|\theta)+1)}\big(D(\ell,\,\ell\lambda_1) * \cdots * D(\ell,\,\ell\lambda_p) * D(m-k-1,\,(m-k-1)\theta)\big) \\ &\to D(\ell,\,\ell\lambda_1) * \cdots * D(\ell,\,\ell\lambda_p) * D(k,\,k\theta) \end{aligned}$$

$$\to D(\ell,\, \ell\lambda_1) * \cdots * D(\ell,\, \ell\lambda_p) * D(m,\, k\theta) \to 0$$

and

$$\begin{aligned}0 &\to \tau_{(2k-m+1)((\lambda|\theta)+1)}\, D\big(t_{w_0\lambda}(\ell-m+k+1)\Lambda_0,\, t_{w_0(\lambda+\theta)}(m-k-1)\Lambda_0\big)\\ &\to D\big(t_{w_0\lambda}(\ell-k)\Lambda_0,\, t_{w_0(\lambda+\theta)}k\Lambda_0\big)\\ &\to D\big(t_{w_0\lambda}(\ell-m)\Lambda_0,\, t_{w_0(\lambda+\theta)}(m\Lambda_0+(m-k)\theta)\big) \to 0.\end{aligned}$$

(3) *If $\ell \le 2k$ and $k \le m < \ell$, then there exist short exact sequences of $\mathfrak{g}[t]$-modules,*

$$\begin{aligned}0 &\to \tau_{(2k-\ell+1)((\lambda|\theta)+1)}\big(D(\ell,\, \ell\lambda_1) * \cdots * D(\ell,\, \ell\lambda_p) * D(\ell+m-2k-1,\, (\ell-k-1)\theta)\big)\\ &\to D(\ell,\, \ell\lambda_1) * \cdots * D(\ell,\, \ell\lambda_p) * D(m,\, k\theta)\\ &\to D(\ell,\, \ell\lambda_1) * \cdots * D(\ell,\, \ell\lambda_p) * D(\ell,\, k\theta) \to 0\end{aligned}$$

and

$$\begin{aligned}0 &\to \tau_{(2k-\ell+1)((\lambda|\theta)+1)}\, D\big(t_{w_0\lambda}(2k-m+1)\Lambda_0,\, t_{w_0(\lambda+\theta)}((\ell+m-2k-1)\Lambda_0+(m-k)\theta)\big)\\ &\to D\big(t_{w_0\lambda}(\ell-m)\Lambda_0,\, t_{w_0(\lambda+\theta)}(m\Lambda_0+(m-k)\theta)\big)\\ &\to D\big(t_{w_0(\lambda+\theta)}(\ell\Lambda_0+(\ell-k)\theta)\big) \to 0.\end{aligned}$$

The following corollary gives a surjective morphism between two fusion products of Demazure modules.

Corollary 3.5. *Let $(\ell \ge m \ge k)$ and $(\ell' \ge m' \ge k')$ be two sequences of positive integers with $k' \le k$. Let $\lambda_1, \dots, \lambda_p$ and $\lambda'_1, \dots, \lambda'_p$ be two sequences of elements in L^+. Denote $\lambda = \sum_{i=1}^p \lambda_i$ and $\lambda' = \sum_{i=1}^p \lambda'_i$. Suppose that*

(a) $\ell\lambda + k\theta = \ell'\lambda' + k'\theta$,

(b) $(\lambda|\alpha) \ge (\lambda'|\alpha), \quad \forall\, \alpha \in R^+$.

If $(\lambda|\theta) = (\lambda'|\theta)$, then we further assume the following holds;

(c) *if $m \ge 2k$, then $m' \ge 2k'$,*

(d) *if $m \le 2k$ and $m' \le 2k'$, then $2k - m \ge 2k' - m'$.*

Then there exist surjective morphisms of $\mathfrak{g}[t]$-modules,

$$\mathbf{V}^{\ell,\, \ell\lambda}_{m,\, k\theta} \twoheadrightarrow \mathbf{V}^{\ell',\, \ell'\lambda'}_{m',\, k'\theta} \tag{3.9}$$

and

$$D(\ell,\, \ell\lambda_1) * \cdots * D(\ell,\, \ell\lambda_p) * D(m,\, k\theta) \twoheadrightarrow D(\ell',\, \ell'\lambda'_1) * \cdots * D(\ell',\, \ell'\lambda'_p) * D(m',\, k'\theta). \tag{3.10}$$

Proof. To prove (3.9), we show that the generator $v' = v^{\ell',\, \ell'\lambda'}_{m',\, k'\theta}$ of $\mathbf{V}^{\ell',\, \ell'\lambda'}_{m',\, k'\theta}$ satisfies the defining relations of $\mathbf{V}^{\ell,\, \ell\lambda}_{m,\, k\theta}$. The relations (3.1) are clear from hypothesis (a). It is easy to see that the relations (3.2) follow by using hypothesis (b). We prove the relations (3.3) by considering two cases. First suppose that $(\lambda|\alpha) > (\lambda'|\alpha)$. Then, they follow from the hypothesis $k' \le k$, by using the relations

$$\big(x^-_\alpha \otimes t^{(\lambda'|\alpha)+1}\big)\, v' = 0, \quad \text{if } \alpha \ne \theta, \qquad \big(x^-_\theta \otimes t^{(\lambda'|\theta)+1}\big)^{k'+1}\, v' = 0,$$

and

$$\left(x_\theta^- \otimes t^{(\lambda'|\theta)+2}\right) v' = 0. \tag{3.11}$$

In the second case, using hypothesis (b), we have $(\lambda|\alpha) = (\lambda'|\alpha)$. Then, they follow from the hypothesis $k' \leq k$ and $(\lambda|\theta) \geq (\lambda'|\theta)$, by using (3.11).

We prove the remaining relations (3.4) and (3.5) by considering two cases. First suppose that $(\lambda|\theta) > (\lambda'|\theta)$. Then, they are immediate from (3.11). In the second case, we have $(\lambda|\theta) = (\lambda'|\theta)$. Then, the relation (3.4) (resp. (3.5)) in this case is clear from hypothesis (d) (resp. hypothesis (c)). Hence (3.9). The proof of (3.10) is immediate from (3.9) by using Theorem 3.3. □

The next corollary, which gives a surjective morphism between two tensor products of Demazure modules, may be viewed as a generalization of the Schur positivity [1].

Corollary 3.6. *Let $(\ell_1 \geq \ell_2 \geq \cdots \geq \ell_p \geq 0)$ and $(m_1 \geq m_2 \geq \cdots \geq m_p \geq 0)$ be two partitions of a positive integer. Suppose that $\ell_i + \cdots + \ell_p \geq m_i + \cdots + m_p$ for each $1 \leq i \leq p$. Then there exists a surjective morphism of $\mathfrak{g}$-modules,*

$$D(\ell_1,\, \ell_1\theta) \otimes D(\ell_2,\, \ell_2\theta) \otimes \cdots \otimes D(\ell_p,\, \ell_p\theta) \twoheadrightarrow D(m_1,\, m_1\theta) \otimes D(m_2,\, m_2\theta) \otimes \cdots \otimes D(m_p,\, m_p\theta).$$

Proof. It is easy to see by using similar arguments of the proof of [1, Theorem 1 (ii)] that it is enough to prove it for $p = 2$. The proof in $p = 2$ case follows from Corollary 3.5. □

4. Proof of the main results

The main goal of this section is to prove Theorems 3.2 and 3.3. The proof uses a result from [3] and the character of generalized Demazure modules.

4.1. In this subsection, we give the $\mathfrak{g}$-module decomposition for the modules $D(\ell,\, k\theta)$, $\ell \geq k$.

Proposition 4.1. *Let $k, \ell \in \mathbb{N}$ be such that $k \leq \ell \leq 2k$. For $i \in \mathbb{Z}_{\geq 0}$, the subspace of $D(\ell,\, k\theta)$ of grade i is given by*

$$D(\ell, k\theta)[i] = \begin{cases} \mathbf{U}(\mathfrak{g})(x_\theta^- \otimes t)^i\, w_{\ell,\, k\theta} \cong_{\mathfrak{g}} V((k-i)\theta), & 0 \leq i \leq 2k-\ell, \\ \{0\}, & i > 2k-\ell. \end{cases}$$

In particular,

$$D(\ell, k\theta) \cong_{\mathfrak{g}} V(k\theta) \oplus V((k-1)\theta) \oplus \cdots \oplus V((\ell-k)\theta).$$

Proof. We prove this by considering the Demazure module corresponding to the module $D(\ell, k\theta)$. Observe that the condition $k \leq \ell \leq 2k$ is equivalent to $\ell\Lambda_0 + (\ell-k)\theta \in \widehat{P}^+$, and

$$t_{w_0(\theta)}(\ell\Lambda_0 + (\ell-k)\theta) = \ell\Lambda_0 + w_0(k\theta) - (2k-\ell)\delta.$$

Hence by Proposition 2.2, we get

$$D(\ell, k\theta) \cong_{\mathfrak{g}[t]} D(t_{w_0(\theta)}(\ell\Lambda_0 + (\ell-k)\theta)). \tag{4.1}$$

Under this isomorphism, the generator $w_{\ell,\, k\theta}$ of $D(\ell,\, k\theta)$ maps to a non-zero element v of the weight space of $D(t_{w_0(\theta)}(\ell\Lambda_0 + (\ell-k)\theta))$ of weight $\ell\Lambda_0 + k\theta - (2k-\ell)\delta$. Considering the $\mathfrak{sl}_2$ copy associated to the real root $\theta - \delta$, it follows from the standard $\mathfrak{sl}_2$ arguments that

$$(x_\theta^- \otimes t)^{2k-\ell}\, v \neq 0,$$

since $(\ell\Lambda_0 + k\theta - (2k-\ell)\delta|\theta-\delta) = 2k-\ell \geq 0$. In particular,

$$(x_\theta^- \otimes t)^{2k-\ell}\, w_{\ell,\,k\theta} \neq 0.$$

The proof now follows by using the following relations which hold in the module $D(\ell, k\theta)$

$$(x_\alpha^- \otimes t)\, w_{\ell,\,k\theta} = 0,\ \forall\ \alpha \in R^+ \setminus \{\theta\}, \qquad (x_\theta^- \otimes t^2)\, w_{\ell,\,k\theta} = 0, \qquad (x_\theta^- \otimes t)^{2k-\ell+1}\, w_{\ell,\,k\theta} = 0,$$

and

$$\mathfrak{n}^+ (x_\theta^- \otimes t)^i\, w_{\ell,\,k\theta} = 0, \quad \forall\ i \in \mathbb{Z}_{\geq 0}.$$

□

We record below an easy fact, for later use.

$$D(\ell, k\theta) \cong_{\mathfrak{g}[t]} D(w_0(\ell\Lambda_0 + k\theta)) \cong_{\mathfrak{g}[t]} \mathrm{ev}_0\, V(k\theta), \quad \forall\, \ell \geq 2k. \tag{4.2}$$

4.2. We now recall the result [3, Theorem 1] for $\lambda^0 = k\theta$. Its proof uses the conclusion of [3, Proposition 3.5] when $\lambda = \lambda^0$, due to which it has some constrains in its hypothesis (see [3, Remark 3.4]). We observe that

$$t_\theta(\ell\Lambda_0 + w_0 k\theta) = \ell\Lambda_0 + (\ell-k)\theta + (2k-\ell)\delta \in \widehat{P}^+, \quad \text{if } k \leq \ell \leq 2k,$$

and

$$w_0(\ell\Lambda_0 + w_0 k\theta) \in \widehat{P}^+, \quad \text{if } \ell \geq 2k.$$

Hence the conclusion of [3, Proposition 3.5] when $\lambda = k\theta$ is satisfied, so we can remove the constrains in the hypothesis of [3, Theorem 1] when $\lambda^0 = k\theta$.

Theorem 4.2. *[3] Let $k \in \mathbb{Z}_{\geq 0}$ and $\ell \in \mathbb{N}$ be such that $\ell \geq k$. Let $\lambda_1, \ldots, \lambda_p$ be a sequence of elements of L^+. Then we have the following isomorphism of $\mathfrak{g}[t]$-modules,*

$$D(\ell,\, \ell\lambda_1) * \cdots * D(\ell,\, \ell\lambda_p) * D(\ell,\, k\theta) \cong D(\ell,\, \ell(\lambda_1 + \cdots + \lambda_p) + k\theta).$$

Under this isomorphism, the generator $w_{\ell,\,\ell\lambda_1} * \cdots * w_{\ell,\,\ell\lambda_p} * w_{\ell,\,k\theta}$ maps to the generator $w_{\ell,\,\ell(\lambda_1+\cdots+\lambda_p)+k\theta}$.

4.3. In this subsection, we prove that for $\ell \geq k$, the module $\mathbf{V}_{\ell,\,k\theta}^{\ell,\,\ell\lambda}$ is isomorphic to the Demazure module $D(\ell,\, \ell\lambda + k\theta)$.

Lemma 4.3. *Let $k, \ell, m \in \mathbb{N}$ be such that $\ell \geq m \geq k$ and $\lambda_1, \ldots, \lambda_p \in L^+$. Denote $\lambda = \lambda_1 + \cdots + \lambda_p$ and $w = w_{\ell,\,\ell\lambda_1} * \cdots * w_{\ell,\,\ell\lambda_p} * w_{m,\,k\theta}$. Then the following relations hold in the module $D(\ell,\, \ell\lambda_1) * \cdots * D(\ell,\, \ell\lambda_p) * D(m,\, k\theta)$*

1. $\left(x_\alpha^- \otimes t^{(\lambda+\theta|\alpha)}\right) w = 0, \quad \forall\ \alpha \in R^+,$
2. $\left(x_\alpha^- \otimes t^{(\lambda|\alpha)}\right)^{\langle (k-i)\theta,\, \alpha^\vee\rangle + 1} \left(x_\theta^- \otimes t^{(\lambda|\theta)+1}\right)^i w = 0, \quad \forall\ \alpha \in R^+,\ \ 0 \leq i \leq k,$
3. $\left(x_\theta^- \otimes t^{(\lambda|\theta)+1}\right)^{2k-m+1} w = 0, \quad$ *if* $m \leq 2k,$
4. $\left(x_\theta^- \otimes t^{(\lambda|\theta)+1}\right) w = 0, \quad$ *if* $m \geq 2k.$

Proof. Let $z_1,\dots,z_p,z_{p+1}$ be the distinct complex numbers which define the fusion product. In the corresponding tensor product, we have

$$\begin{aligned}&\left(x_\alpha^-\otimes(t-z_1)^{(\lambda_1|\alpha)}\cdots(t-z_p)^{(\lambda_p|\alpha)}(t-z_{p+1})^{(\theta|\alpha)}\right)(w_{\ell,\,\ell\lambda_1}\otimes\cdots\otimes w_{\ell,\,\ell\lambda_p}\otimes w_{m,\,k\theta})\\&=\sum_{j=1}^{p+1}\left(w_{\ell,\,\ell\lambda_1}\otimes\cdots\otimes(x_\alpha^-\otimes t^{(\lambda_j|\alpha)}f_j(t)\,w_{\ell,\,\ell\lambda_j})\otimes\cdots w_{\ell,\,\ell\lambda_p}\otimes w_{m,\,k\theta}\right)=0,\end{aligned}$$

where $f_j(t)=\prod_{i\neq j}(t+z_j-z_i)^{(\lambda_i|\alpha)}$, $\lambda_{p+1}=\theta$, and the last equality follows by using the relations

$$(x_\alpha^-\otimes t^{(\lambda_j|\alpha)})\,w_{\ell,\,\ell\lambda_j}=0,\ \forall\ 1\le j\le p\qquad\text{and}\qquad(x_\alpha^-\otimes t^{(\theta|\alpha)})\,w_{m,\,k\theta}=0.$$

Now part (1) is immediate. The proof of part (2) (resp. part (3), part (4)) is identical by using the relation

$$(x_\alpha^-\otimes 1)^{\langle(k-i)\theta,\,\alpha^\vee\rangle+1}(x_\theta^-\otimes t)^i\,w_{m,\,k\theta}=0\quad\text{(follows from Proposition 4.1)}$$

$$\text{(resp. }(x_\theta^-\otimes t)^{2k-m+1}\,w_{m,\,k\theta}=0\quad(\text{since } m\le 2k),\qquad(x_\theta^-\otimes t)\,w_{m,\,k\theta}=0\quad(\text{since } m\ge 2k)),$$

and we omit the details. □

The following proposition gives explicit defining relations for the modules $D(\ell,\ell\lambda+k\theta)$, $\ell\ge k$.

Proposition 4.4. *Let $k,\ell\in\mathbb{N}$ be such that $\ell\ge k$ and $\lambda\in L^+$. Let $w=w_{\ell,\,\ell\lambda+k\theta}$ be the generator of the module $D(\ell,\ell\lambda+k\theta)$. Then:*

(1) The following are the defining relations for the module $D(\ell,\ell\lambda+k\theta)$

$$(x_i^+\otimes t^s)\,w=0,\ \ (\alpha_i^\vee\otimes t^s)\,w=\delta_{s,0}\langle\ell\lambda+k\theta,\,\alpha_i^\vee\rangle\,w,\ \ \left(x_i^-\otimes 1\right)^{\langle\ell\lambda+k\theta,\,\alpha_i^\vee\rangle+1}w=0,\quad\forall\ s\ge0, i\in I,$$

$$\left(x_\alpha^-\otimes t^{(\lambda+\theta|\alpha)}\right)w=0,\quad\forall\,\alpha\in R^+,$$

$$\left(x_\alpha^-\otimes t^{(\lambda|\alpha)}\right)^{\langle k\theta,\,\alpha^\vee\rangle+1}w=0,\quad\forall\,\alpha\in R^+,$$

$$\left(x_\theta^-\otimes t^{(\lambda|\theta)+1}\right)^{2k-\ell+1}w=0,\quad\text{if } \ell\le 2k,$$

$$\left(x_\theta^-\otimes t^{(\lambda|\theta)+1}\right)w=0,\quad\text{if } \ell\ge 2k.$$

(2) The following relations also hold in the module $D(\ell,\ell\lambda+k\theta)$

$$\left(x_\alpha^-\otimes t^{(\lambda|\alpha)}\right)^{\langle(k-i)\theta,\,\alpha^\vee\rangle+1}\left(x_\theta^-\otimes t^{(\lambda|\theta)+1}\right)^i w=0,\quad\forall\ \alpha\in R^+,\ 1\le i\le k.$$

Proof. For $\alpha\in R^+$ with $\langle\ell\lambda+k\theta,\alpha^\vee\rangle>0$, we first write down the positive integers s_α,m_α such that

$$\langle\ell\lambda+k\theta,\alpha^\vee\rangle=(s_\alpha-1)d_\alpha\ell+m_\alpha,\quad 0<m_\alpha\le d_\alpha\ell.$$

If $(\theta|\alpha)=0$ (resp. $(\theta|\alpha)=1$), then $s_\alpha=(\lambda|\alpha)$ (resp. $s_\alpha=(\lambda|\alpha)+1$) and $m_\alpha=d_\alpha\ell$ (resp. $m_\alpha=d_\alpha k$). We now consider the remaining case when $\alpha=\theta$. If $\ell<2k$ (resp. $\ell\ge 2k$), then $s_\theta=(\lambda|\theta)+2$ (resp. $s_\theta=(\lambda|\theta)+1$) and $m_\theta=2k-\ell$ (resp. $m_\theta=2k$). The proof now follows from the definition of the module $D(\ell,\,\ell\lambda+k\theta)$, by using Theorem 4.2 and Lemma 4.3. □

Proposition 4.5. *Given $k,\ell\in\mathbb{N}$ such that $\ell\ge k$ and $\lambda\in L^+$, we have the following isomorphism of $\mathfrak{g}[t]$-modules,*

$$\mathbf{V}^{\ell,\,\ell\lambda}_{\ell,\,k\theta}\cong D(\ell,\ell\lambda+k\theta).$$

Proof. The proof is immediate from Proposition 4.4. □

4.4. We now establish the existence of the maps ϕ_m^+, ϕ_1^- and ϕ_2^- from Theorem 3.2.

The following proposition, which gives the existence of ϕ_m^+, is trivially checked.

Proposition 4.6. *Let $k, \ell, m \in \mathbb{N}$ be such that $\ell \geq m \geq k$ and $\lambda \in L^+$. Then the map*

$$\phi_m^+ : \mathbf{V}_{m,\,k\theta}^{\ell,\,\ell\lambda} \to \mathbf{V}_{\ell,\,k\theta}^{\ell,\,\ell\lambda} \quad \textit{such that} \quad \phi_m^+(v_{m,\,k\theta}^{\ell,\,\ell\lambda}) = v_{\ell,\,k\theta}^{\ell,\,\ell\lambda},$$

is a surjective morphism of $\mathfrak{g}[t]$-modules with

$$\ker \phi_m^+ = \begin{cases} \mathbf{U}(\mathfrak{g}[t]) \left(x_\theta^- \otimes t^{(\lambda|\theta)+1}\right)^{2k-\ell+1} v_{m,\,k\theta}^{\ell,\,\ell\lambda}, & \ell \leq 2k, \\ \mathbf{U}(\mathfrak{g}[t]) \left(x_\theta^- \otimes t^{(\lambda|\theta)+1}\right) v_{m,\,k\theta}^{\ell,\,\ell\lambda}, & \ell \geq 2k. \end{cases}$$

Lemma 4.7. *Given $\lambda \in L^+$, we have the following:*

(1) For $k, \ell, m \in \mathbb{N}$ such that $k \leq m < \ell \leq 2k$, the following relations hold in the module $\mathbf{V}_{m,\,k\theta}^{\ell,\,\ell\lambda}$.
   - *(a) $(x_\alpha^+ \otimes t^s)(x_\theta^- \otimes t^{(\lambda|\theta)+1})^{2k-\ell+1} v_{m,\,k\theta}^{\ell,\,\ell\lambda} = 0, \quad \forall\ s \geq 0,\ \alpha \in R^+$,*
   - *(b) $(x_\alpha^- \otimes 1)^{\langle \ell\lambda+(\ell-k-1)\theta,\,\alpha^\vee\rangle+1}(x_\theta^- \otimes t^{(\lambda|\theta)+1})^{2k-\ell+1} v_{m,\,k\theta}^{\ell,\,\ell\lambda} = 0, \quad \forall\ \alpha \in R^+$.*

(2) For $k, \ell \in \mathbb{N}$ such that $\ell \geq k$, the following relations hold in the module $\mathbf{V}_{k,\,k\theta}^{\ell,\,\ell\lambda}$.
   - *(a) $(x_\alpha^+ \otimes t^s)(x_\theta^- \otimes t^{(\lambda|\theta)+1}) v_{k,\,k\theta}^{\ell,\,\ell\lambda} = 0, \quad \forall\ s \geq 0,\ \alpha \in R^+$,*
   - *(b) $(x_\alpha^- \otimes 1)^{\langle \ell\lambda+(k-1)\theta,\,\alpha^\vee\rangle+1}(x_\theta^- \otimes t^{(\lambda|\theta)+1}) v_{k,\,k\theta}^{\ell,\,\ell\lambda} = 0, \quad \forall\ \alpha \in R^+$.*

Proof. Set $v = v_{m,\,k\theta}^{\ell,\,\ell\lambda}$. To prove part (1a), since $(x_\alpha^+ \otimes t^s)\, v = 0$, it is enough to prove that

$$[x_\alpha^+ \otimes t^s, (x_\theta^- \otimes t^{(\lambda|\theta)+1})^{2k-\ell+1}]\, v = 0.$$

This means to show that

$$\sum_{j=0}^{2k-\ell} \left(x_\theta^- \otimes t^{(\lambda|\theta)+1}\right)^j \left([x_\alpha^+, x_\theta^-] \otimes t^{(\lambda|\theta)+s+1}\right) \left(x_\theta^- \otimes t^{(\lambda|\theta)+1}\right)^{2k-\ell-j} v = 0. \tag{4.3}$$

If $(\theta|\alpha) = 0$, then $[x_\alpha^+, x_\theta^-] = 0$ and hence (4.3). If $(\theta|\alpha) = 1$, then $\theta - \alpha \in R^+$ and $[x_\alpha^+, x_\theta^-]$ is a non-zero scalar multiple of $x_{\theta-\alpha}^-$. The proof of part (4.3) in this case follows now from the relation $(x_{\theta-\alpha}^- \otimes t^{(\lambda+\theta|\theta-\alpha)})\, v = 0$. In the remaining case when $\alpha = \theta$, it follows from the relations

$$(x_\theta^- \otimes t^{(\lambda|\theta)+s'+1})\, v = 0 \quad \text{and} \quad (\theta^\vee \otimes t^{s'})\, v = 0, \qquad \text{for every } s' \geq 1.$$

We now prove part (1b). We observe that $(x_\theta^- \otimes t^{(\lambda|\theta)+1})^{2k-\ell+1}\, v$ is an element of weight $\ell\lambda + (\ell-k-1)\theta$. Considering the $\mathfrak{sl}_2$ copy spanned by $x_\alpha^+ \otimes 1, x_\alpha^- \otimes 1$, and $\alpha^\vee \otimes 1$, the proof of part (1b) follows by standard $\mathfrak{sl}_2$ calculations, using part (1a). The proof of part (2) is similar and we omit the details. □

The next proposition gives the existence of ϕ_1^- and ϕ_2^-.

Proposition 4.8. *Let $k, \ell, m \in \mathbb{N}$ and $\lambda \in L^+$. Denote $m' = \ell + m - 2k - 1$ and $k' = \ell - k - 1$.*

(1) If $\ell \leq 2k$ and $k \leq m < \ell$, then the map $\phi_2^- : \tau_{(2k-\ell+1)((\lambda|\theta)+1)} \mathbf{V}_{m',\,k'\theta}^{\ell,\,\ell\lambda} \to \ker \phi_m^+$ which takes $v_{m',\,k'\theta}^{\ell,\,\ell\lambda} \mapsto \left(x_\theta^- \otimes t^{(\lambda|\theta)+1}\right)^{2k-\ell+1} v_{m,\,k\theta}^{\ell,\,\ell\lambda}$ is a surjective morphism of $\mathfrak{g}[t]$-modules.

(2) *If $\ell \geq 2k$, then the map $\phi_1^- : \tau_{(\lambda|\theta)+1}\mathbf{V}^{\ell,\,\ell\lambda}_{k-1,\,(k-1)\theta} \to \ker \phi_k^+$ which takes $v^{\ell,\,\ell\lambda}_{k-1,\,(k-1)\theta} \mapsto \left(x_\theta^- \otimes t^{(\lambda|\theta)+1}\right) v^{\ell,\,\ell\lambda}_{k,\,k\theta}$ is a surjective morphism of $\mathfrak{g}[t]$-modules.*

Proof. Set $v = v^{\ell,\,\ell\lambda}_{m,\,k\theta}$. To prove part (1), we need to show that $\left(x_\theta^- \otimes t^{(\lambda|\theta)+1}\right)^{2k-\ell+1} v$ satisfies the defining relations of the module $\mathbf{V}^{\ell,\,\ell\lambda}_{m',\,k'\theta}$. Since $m' \leq 2k'$, using Lemma 4.7 (1) and the relations (3.2), we only need to show the following:

$$\left(x_\alpha^- \otimes t^{(\lambda|\alpha)}\right)^{\langle (k'-i')\theta,\,\alpha^\vee\rangle+1}\left(x_\theta^- \otimes t^{(\lambda|\theta)+1}\right)^{i'+2k-\ell+1} v = 0, \quad \forall\ \alpha \in R^+,\ \ 0 \leq i' \leq k', \tag{4.4}$$

$$\left(x_\theta^- \otimes t^{(\lambda|\theta)+1}\right)^{(2k'-m'+1)+(2k-\ell+1)} v = 0. \tag{4.5}$$

Since $k' - i' = k - (i' + 2k - \ell + 1)$, the relations (4.4) follow from the relations (3.3). The relation (4.5) is same as the relation (3.4). Hence part (1). The proof of part (2) is similar and we omit the details. $\square$

4.5. Using (4.2), Theorem 4.2, and Proposition 4.5, the existence of surjective map ϕ_m^+ and the maps ϕ_1^- and ϕ_2^- give the following inequalities:

$$\dim \mathbf{V}^{\ell,\,\ell\lambda}_{k,\,k\theta} \leq \dim \mathbf{V}^{\ell,\,\ell\lambda}_{k-1,\,(k-1)\theta} + \dim V(k\theta) \prod_{j=1}^{p} \dim D(\ell,\,\ell\lambda_j),\ \ \forall\ \ell \geq 2k, \tag{4.6}$$

$$\dim \mathbf{V}^{\ell,\,\ell\lambda}_{m,\,k\theta} \leq \dim \mathbf{V}^{\ell,\,\ell\lambda}_{\ell+m-2k-1,\,(\ell-k-1)\theta} + \dim D(\ell,\,k\theta) \prod_{j=1}^{p} \dim D(\ell,\,\ell\lambda_j),\ \ \forall\ k \leq m < \ell \leq 2k, \tag{4.7}$$

for every $\lambda_1, \ldots, \lambda_p \in L^+$ with $\lambda_1 + \cdots + \lambda_p = \lambda$.

The following proposition is useful in getting the reverse inequalities.

Proposition 4.9. *Let $k, \ell, m \in \mathbb{N}$ be such that $\ell \geq m \geq k$. Let $\lambda_1, \ldots, \lambda_p \in L^+$ and denote $\lambda = \lambda_1 + \cdots + \lambda_p$. Then the assignment $v^{\ell,\,\ell\lambda}_{m,\,k\theta} \mapsto w_{\ell,\,\ell\lambda_1} * \cdots * w_{\ell,\,\ell\lambda_p} * w_{m,\,k\theta}$ defines a surjective morphism of $\mathfrak{g}[t]$-modules*

$$\mathbf{V}^{\ell,\,\ell\lambda}_{m,\,k\theta} \twoheadrightarrow D(\ell,\,\ell\lambda_1) * \cdots * D(\ell,\,\ell\lambda_p) * D(m,\,k\theta).$$

In particular,

$$\dim \mathbf{V}^{\ell,\,\ell\lambda}_{m,\,k\theta} \geq \dim D(m,\,k\theta) \prod_{j=1}^{p} \dim D(\ell,\,\ell\lambda_j).$$

Proof. The proof follows from Lemma 4.3. $\square$

4.6. **Proof of Theorem 3.2.** Let $\lambda_1, \ldots, \lambda_p \in L^+$ be such that $\lambda_1 + \cdots + \lambda_p = \lambda$.

(1) To prove part (1), we prove that

$$\mathbf{V}^{\ell,\,\ell\lambda}_{k,\,k\theta} \cong_{\mathfrak{g}[t]} D(\ell,\,\ell\lambda_1) * \cdots * D(\ell,\,\ell\lambda_p) * D(k,\,k\theta), \quad \text{for } \ell \geq 2k. \tag{4.8}$$

We proceed by induction on k. In the case $k=1$, by using (3.8) and Theorem 4.2, we have $\dim \mathbf{V}^{\ell,\,\ell\lambda}_{0,\,0\theta} = \prod_{j=1}^{p} \dim D(\ell,\,\ell\lambda_j)$. Now (4.6) gives,

$$\dim \mathbf{V}^{\ell,\,\ell\lambda}_{1,\,1\theta} \leq (\dim V(\theta)+1) \prod_{j=1}^{p} \dim D(\ell,\,\ell\lambda_j). \tag{4.9}$$

Using Proposition 4.1, we have

$$\dim D(1,\,\theta) = \dim V(\theta) + 1. \tag{4.10}$$

We get (4.8) in this case from Proposition 4.9, by using (4.9) and (4.10).

Now suppose $k \geq 2$. By the induction hypothesis, we have

$$\dim \mathbf{V}^{\ell,\,\ell\lambda}_{k-1,\,(k-1)\theta} = \dim D(k-1,\,(k-1)\theta) \prod_{j=1}^{p} \dim D(\ell,\,\ell\lambda_j).$$

Substituting this into (4.6), we get

$$\dim \mathbf{V}^{\ell,\,\ell\lambda}_{k,\,k\theta} \leq \big(\dim V(k\theta) + \dim D(k-1,\,(k-1)\theta)\big) \prod_{j=1}^{p} \dim D(\ell,\,\ell\lambda_j).$$

Now the proof of (4.8) follows from Proposition 4.9, by using Proposition 4.1.

(2) The existence of ψ^- such that

$$\psi^-\big(v^{\ell,\,\ell\lambda}_{m-k-1,\,(m-k-1)\theta}\big) = \big(x^-_\theta \otimes t^{(\lambda|\theta)+1}\big)^{2k-m+1}\, v^{\ell,\,\ell\lambda}_{k,\,k\theta},$$

and ψ^- is injective follow from repeated applications of the injective morphism ϕ^-_1 from part (1) for various values of k. It is easy to see that the assignment $v^{\ell,\,\ell\lambda}_{k,\,k\theta} \mapsto v^{\ell,\,\ell\lambda}_{m,\,k\theta}$ gives the existence of ψ^+. Clearly ψ^+ is surjective and

$$\ker \psi^+ = \mathbf{U}(\mathfrak{g}[t])\,\big(x^-_\theta \otimes t^{(\lambda|\theta)+1}\big)^{2k-m+1}\, v^{\ell,\,\ell\lambda}_{k,\,k\theta}.$$

Now we observe that $\operatorname{img}\psi^- = \ker\psi^+$. This completes the proof of part (2), and it gives,

$$\dim\, \mathbf{V}^{\ell,\,\ell\lambda}_{m,\,k\theta} = \dim\, \mathbf{V}^{\ell,\,\ell\lambda}_{k,\,k\theta} - \dim\, \mathbf{V}^{\ell,\,\ell\lambda}_{m-k-1,\,(m-k-1)\theta}.$$

Since $\ell \geq 2k > 2(m-k-1)$, using (4.8), the right hand side of the last equation becomes

$$\big(\dim D(k,\,k\theta) - \dim D(m-k-1,\,(m-k-1)\theta)\big) \prod_{j=1}^{p} \dim D(\ell,\,\ell\lambda_j).$$

Now using Propositions 4.1 and 4.9, we get

$$\mathbf{V}^{\ell,\,\ell\lambda}_{m,\,k\theta} \cong_{\mathfrak{g}[t]} D(\ell,\,\ell\lambda_1) * \cdots * D(\ell,\,\ell\lambda_p) * D(m,\,k\theta), \quad \text{for } k < m \leq 2k \leq \ell. \tag{4.11}$$

(3) Let $k' = \ell - k - 1$ and $m' = \ell + m - 2k - 1$. Since $\ell \geq 2k'$ and $k' \leq m' \leq 2k'$, using (4.8) and (4.11), we get

$$\dim \mathbf{V}^{\ell,\,\ell\lambda}_{m',\,k'\theta} = \dim D(m',\,k'\theta) \prod_{j=1}^{p} \dim D(\ell,\,\ell\lambda_j).$$

Substituting this into (4.7), we get

$$\dim \mathbf{V}^{\ell,\,\ell\lambda}_{m,\,k\theta} \le \big(\dim D(m',\,k'\theta) + \dim D(\ell,\,k\theta)\big)\prod_{j=1}^{p}\dim D(\ell,\,\ell\lambda_j).$$

Now Proposition 4.1 gives,

$$\dim \mathbf{V}^{\ell,\,\ell\lambda}_{m,\,k\theta} \le \dim D(m,\,k\theta)\prod_{j=1}^{p}\dim D(\ell,\,\ell\lambda_j).$$

Using Proposition 4.9, we get part (3), and

$$\mathbf{V}^{\ell,\,\ell\lambda}_{m,\,k\theta} \cong_{\mathfrak{g}[t]} D(\ell,\,\ell\lambda_1) * \cdots * D(\ell,\,\ell\lambda_p) * D(m,\,k\theta), \quad \text{for } k \le m < \ell \le 2k. \tag{4.12}$$

□

4.7. Although the following proposition seems to be well known, we give a proof here for the sake of completeness.

Proposition 4.10. *For $1 \le j \le p$, let $\Lambda^j \in \widehat{P}^+, \xi_j \in \widehat{W}\Lambda^j$ such that $\langle \xi_j,\, \alpha_i^\vee\rangle \le 0, \forall\, i \in I$, and $v_{w_0\xi_j} \in V(\Lambda^j)_{w_0\xi_j}$. Then*

$$D\big(\xi_1,\dots,\xi_p\big) = \mathbf{U}(\mathfrak{g}[t])\,(v_{w_0\xi_1}\otimes\cdots\otimes v_{w_0\xi_p}).$$

Proof. We first prove that $v_{w_0\xi_1}\otimes\cdots\otimes v_{w_0\xi_p} \in D\big(\xi_1,\dots,\xi_p\big)$. Let $v_{\xi_j} \in V(\Lambda^j)_{\xi_j}$, for $1 \le j \le p$, and set $v = v_{\xi_1}\otimes\cdots\otimes v_{\xi_p}$. Since $\mathfrak{n}^-\,v = 0$, the $\mathfrak{g}$-submodule $\mathbf{U}(\mathfrak{g})\,v \subseteq D\big(\xi_1,\dots,\xi_p\big)$ is isomorphic to the irreducible highest weight $\mathfrak{g}$-module $V\big(w_0(\xi_1+\cdots+\xi_p)|_{\mathfrak{h}}\big)$. Hence there exists a non-zero element $v' \in \mathbf{U}(\mathfrak{n}^+)\,v$ in $D\big(\xi_1,\dots,\xi_p\big)$, whose $\mathfrak{h}$-weight is equal to $w_0(\xi_1+\cdots+\xi_p)|_{\mathfrak{h}}$. Suppose that $\xi_1+\cdots+\xi_p = a\Lambda_0+\beta+b\delta$, for some $a,b\in\mathbb{C}$ and $\beta\in\mathfrak{h}^*$, then the weight of v' is equal to $a\Lambda_0+w_0\beta+b\delta = w_0(\xi_1+\cdots+\xi_p)$. Hence v' is a non-zero constant multiple of $v_{w_0\xi_1}\otimes\cdots\otimes v_{w_0\xi_p}$, since

$$\big(D(\xi_1)\otimes\cdots\otimes D(\xi_p)\big)_{w_0(\xi_1+\cdots+\xi_p)} = \mathbb{C}\,(v_{w_0\xi_1}\otimes\cdots\otimes v_{w_0\xi_p}).$$

Now we have

$$\mathbf{U}(\mathfrak{g}[t])\,(v_{w_0\xi_1}\otimes\cdots\otimes v_{w_0\xi_p}) \subseteq D\big(\xi_1,\dots,\xi_p\big).$$

The reverse containment also follows in the similar way and we omit the details. □

Proposition 4.11. *Let $k,\ell,m \in \mathbb{N}$ be such that $\ell \ge m \ge k$ and $\lambda \in L^+$. Then there exists a morphism of $\mathfrak{g}[t]$-modules*

$$\varphi : \mathbf{V}^{\ell,\,\ell\lambda}_{m,\,k\theta} \to D(\ell-m,\,(\ell-m)\lambda)\otimes D(m,\,m\lambda+k\theta),$$

such that

$$\varphi\big(v^{\ell,\,\ell\lambda}_{m,\,k\theta}\big) = w_{\ell-m,\,(\ell-m)\lambda}\otimes w_{m,\,m\lambda+k\theta} \qquad \text{and} \qquad \operatorname{img}\varphi = \mathbf{U}(\mathfrak{g}[t])\,(w_{\ell-m,\,(\ell-m)\lambda}\otimes w_{m,\,m\lambda+k\theta}).$$

Proof. The proof follows by using Proposition 4.4 and the relations

$$(x_\alpha^-\otimes t^{(\lambda|\alpha)})\,w_{\ell-m,\,(\ell-m)\lambda} = 0, \quad \forall\ \alpha\in R^+,$$

which hold in the module $D(\ell-m,\,(\ell-m)\lambda)$. □

Proposition 4.12. *With hypothesis and notation as in Proposition 4.11, there exist surjective morphisms of $\mathfrak{g}[t]$-modules,*

$$\mathbf{V}_{m,\,k\theta}^{\ell,\,\ell\lambda} \twoheadrightarrow \begin{cases} D\big(t_{w_0\lambda}(\ell-m)\Lambda_0,\, t_{w_0(\lambda+\theta)}(m\Lambda_0+(m-k)\theta)\big), & m\le 2k,\\ D\big(t_{w_0\lambda}(\ell-m)\Lambda_0,\, t_{w_0\lambda}w_0(m\Lambda_0+k\theta)\big), & m\ge 2k.\end{cases}$$

Proof. We observe from Proposition 2.2 that

$$D(\ell-m,\,(\ell-m)\lambda)\cong_{\mathfrak{g}[t]} D(t_{w_0\lambda}(\ell-m)\Lambda_0)$$

and

$$D(m,\,m\lambda+k\theta)\cong_{\mathfrak{g}[t]} \begin{cases} D(t_{w_0(\lambda+\theta)}(m\Lambda_0+(m-k)\theta)), & m\le 2k,\\ D(t_{w_0\lambda}w_0(m\Lambda_0+k\theta)), & m\ge 2k.\end{cases}$$

Now the proof follows from Proposition 4.11, by using Proposition 4.10. □

4.8. In this subsection, we recollect some facts about the Demazure operators and the character of generalized Demazure modules, which are useful in proving Theorem 3.3.

For $0\le i\le n$, the Demazure operator $\mathcal{D}_i$ is a linear operator on $\mathbb{Z}[\widehat{P}]$, and is defined by

$$\mathcal{D}_i(e^\Lambda)=\frac{e^\Lambda-e^{r_i(\Lambda)-\alpha_i}}{1-e^{-\alpha_i}}.$$

For $w\in\widehat{W}$ and a reduced expression $w=r_{i_1}\cdots r_{i_k}$, the Demazure operator $\mathcal{D}_w$ is defined as $\mathcal{D}_w=\mathcal{D}_{i_1}\cdots\mathcal{D}_{i_k}$, and is independent of the choice of reduced expression of w ([9, Corollary 8.2.10]). For $w\in\widehat{W}$ and $\sigma\in\Sigma$, set $\mathcal{D}_{w\sigma}(e^\Lambda)=\mathcal{D}_w(e^{\sigma\Lambda})$. Since $\mathcal{D}_i(e^\delta)=e^\delta$, the operator $\mathcal{D}_w$ descends to $\mathbb{Z}[\widehat{P}]/I_\delta$, for all $w\in\widetilde{W}$.

The following theorem gives the character of generalized Demazure modules in terms of Demazure operators. It is a combination of the results [15, Proposition 2.7 and Corollary 2.8]. The key ingredient in the proof is a result from [11].

Theorem 4.13. *[11, 15] Let $\Lambda^1,\dots,\Lambda^p$ be a sequence of elements of $\widehat{P}^+$. Let $w_1,\dots,w_p$ be a sequence of elements of $\widetilde{W}$, and denote $w_{[1,j]}=w_1\cdots w_j$, $\forall\,1\le j\le p$. If $\ell(w_{[1,p]})=\sum_{j=1}^p\ell(w_j)$, then we have*

$$\begin{aligned}&\mathrm{ch}_{\widehat{\mathfrak{h}}}\, D\left(w_{[1,1]}\Lambda^1, w_{[1,2]}\Lambda^2,\dots,w_{[1,p-1]}\Lambda^{p-1},w_{[1,p]}\Lambda^p\right)\\ &=\mathcal{D}_{w_1}\left(e^{\Lambda^1}\,\mathcal{D}_{w_2}\left(e^{\Lambda^2}\cdots\mathcal{D}_{w_{p-1}}\left(e^{\Lambda^{p-1}}\,\mathcal{D}_{w_p}(e^{\Lambda^p})\right)\cdots\right)\right).\end{aligned}$$

The following theorem may be found in [8, Theorem 3.4] and [12] (see also [9, Theorem 8.2.9] and [3, §4.5]).

Theorem 4.14. *[8, 12] Given $(\ell,\lambda)\in\mathbb{N}\times P^+$, let $w\in\widehat{W},\sigma\in\Sigma$ and $\Lambda\in\widehat{P}^+$ such that*

$$w\sigma\Lambda\equiv w_0\lambda+\ell\Lambda_0 \mod \mathbb{Z}\delta.$$

Then

$$\mathcal{D}_{w\sigma}(e^\Lambda)\equiv e^{\ell\Lambda_0}\mathrm{ch}_{\mathfrak{h}}\, D(\ell,\lambda) \mod I_\delta.$$

The next lemma follows as [6, Lemma 7] (see also [3, §4.4]).

Lemma 4.15. *[6] Let V be a finite-dimensional $\mathfrak{g}$-module. Let $(\ell,\mu)\in\mathbb{N}\times L^+$. Then for $A\in\mathbb{Z}$, we have*

$$\mathcal{D}_{t_{w_0\mu}}(e^{\ell\Lambda_0+A\delta}\operatorname{ch}_{\mathfrak{h}}V)=\mathcal{D}_{t_{w_0\mu}}(e^{\ell\Lambda_0+A\delta})\operatorname{ch}_{\mathfrak{h}}V.$$

4.9. We now prove that the fusion product of Demazure modules of different level as a $\mathfrak{g}$-module is isomorphic to a generalized Demazure module. We further conjecture that they are in fact isomorphic as $\mathfrak{g}[t]$-modules.

Proposition 4.16. *Let $(\ell,\lambda)\in\mathbb{N}\times P^+$. Suppose that there exist $\nu\in P^+, w\in W$, and $\Lambda\in\widehat{P}^+$ such that*

$$t_{-\nu}w\Lambda\equiv w_0\lambda+\ell\Lambda_0 \mod \mathbb{Z}\delta.$$

Let $(\ell_i,\lambda_i)\in\mathbb{N}\times L^+$, for $1\le i\le p$, be such that $\ell_1\ge\cdots\ge\ell_p\ge\ell$. Then we have the following isomorphism as $\mathfrak{g}$-modules:

$$\begin{aligned}&D(\ell_1,\,\ell_1\lambda_1)*\cdots*D(\ell_p,\,\ell_p\lambda_p)*D(\ell,\,\lambda)\\&\cong D\big(t_{w_0\lambda_1}(\ell_1-\ell_2)\Lambda_0,\ldots,t_{w_0(\lambda_1+\cdots+\lambda_p)}(\ell_p-\ell)\Lambda_0,\,t_{w_0(\lambda_1+\cdots+\lambda_p)}\,t_{-\nu}w\Lambda\big).\end{aligned}\tag{4.13}$$

Proof. Since the finite-dimensional $\mathfrak{g}$-modules are determined by their $\mathfrak{h}$-characters, it suffices to show that

$$\begin{aligned}&\operatorname{ch}_{\mathfrak{h}}D\big(t_{w_0\lambda_1}(\ell_1-\ell_2)\Lambda_0,\ldots,\,t_{w_0(\lambda_1+\cdots+\lambda_p)}(\ell_p-\ell)\Lambda_0,\,t_{w_0(\lambda_1+\cdots+\lambda_p)}\,t_{-\nu}w\Lambda\big)\\&=\operatorname{ch}_{\mathfrak{h}}D(\ell_1,\,\ell_1\lambda_1)\cdots\operatorname{ch}_{\mathfrak{h}}D(\ell_p,\,\ell_p\lambda_p)\operatorname{ch}_{\mathfrak{h}}D(\ell,\,\lambda).\end{aligned}\tag{4.14}$$

From Lemma 2.1 and Theorem 4.13, we have

$$\begin{aligned}&\operatorname{ch}_{\widehat{\mathfrak{h}}}D\big(t_{w_0\lambda_1}(\ell_1-\ell_2)\Lambda_0,\ldots,\,t_{w_0(\lambda_1+\cdots+\lambda_p)}(\ell_p-\ell)\Lambda_0,\,t_{w_0(\lambda_1+\cdots+\lambda_p)}\,t_{-\nu}w\Lambda\big)\\&=\mathcal{D}_{t_{w_0\lambda_1}}\big(e^{(\ell_1-\ell_2)\Lambda_0}\,\mathcal{D}_{t_{w_0\lambda_2}}\big(e^{(\ell_2-\ell_3)\Lambda_0}\cdots\mathcal{D}_{t_{w_0\lambda_p}}\big(e^{(\ell_p-\ell)\Lambda_0}\,\mathcal{D}_{t_{-\nu}w}\big(e^{\Lambda}\big)\big)\cdots\big)\big).\end{aligned}$$

Now repeated applications of Theorem 4.14 and Lemma 4.15 give,

$$\begin{aligned}&\operatorname{ch}_{\widehat{\mathfrak{h}}}D\big(t_{w_0\lambda_1}(\ell_1-\ell_2)\Lambda_0,\ldots,\,t_{w_0(\lambda_1+\cdots+\lambda_p)}(\ell_p-\ell)\Lambda_0,\,t_{w_0(\lambda_1+\cdots+\lambda_p)}\,t_{-\nu}w\Lambda\big)\\&\equiv e^{\ell_1\Lambda_0}\operatorname{ch}_{\mathfrak{h}}D(\ell_1,\,\ell_1\lambda_1)\cdots\operatorname{ch}_{\mathfrak{h}}D(\ell_p,\,\ell_p\lambda_p)\operatorname{ch}_{\mathfrak{h}}D(\ell,\,\lambda)\quad \mod I_{\delta}.\end{aligned}$$

Letting $e^{\Lambda_0}\mapsto 1$ and $e^{\delta}\mapsto 1$, we get (4.14). Hence the proposition. □

We conjecture below that the isomorphism (4.13) also holds as $\mathfrak{g}[t]$-modules.

Conjecture 4.17. *Under the hypothesis of Proposition 4.16, we have the isomorphism* (4.13) *of $\mathfrak{g}[t]$-modules.*

Remark 4.18. *This conjecture is proved when $\ell_1=\cdots=\ell_p=\ell$ in [3]. In [14], for $\mathfrak{g}$ simply laced, it is proved when $\lambda_1,\ldots,\lambda_p$, and $\frac{1}{\ell}\lambda$ are fundamental weights. In this paper, we prove a special case of it and also give the defining relations (see Theorem 3.3).*

4.10. **Proof of Theorem 3.3.** We first prove that for $\ell \geq m \geq k$,

$$D(\ell,\, \ell\lambda_1) * \cdots * D(\ell,\, \ell\lambda_p) * D(m,\, k\theta) \cong_{\mathfrak{g}[t]} \mathbf{V}^{\ell,\, \ell\lambda}_{m,\, k\theta}. \tag{4.15}$$

We consider two cases. First, suppose that $\ell \geq 2k$. Then, from (4.8) and (4.11) we obtain (4.15) when $k \leq m \leq 2k$. If $m > 2k$, then since $D(m,\, k\theta) \cong_{\mathfrak{g}[t]} D(2k,\, k\theta)$, it follows from the case $m = 2k$ and using (3.7). In the second case, we have $\ell \leq 2k$. Then, from (4.12) we obtain (4.15) when $\ell \neq m$. If $\ell = m$, then it follows by using Theorem 4.2 and Proposition 4.5.

Next we prove that for $\ell \geq m \geq k$,

$$\mathbf{V}^{\ell,\, \ell\lambda}_{m,\, k\theta} \cong_{\mathfrak{g}[t]} \begin{cases} D\big(t_{w_0\lambda}(\ell - m)\Lambda_0,\, t_{w_0(\lambda+\theta)}(m\Lambda_0 + (m-k)\theta)\big), & m \leq 2k, \\ D\big(t_{w_0\lambda}(\ell - m)\Lambda_0,\, t_{w_0\lambda}w_0(m\Lambda_0 + k\theta)\big), & m \geq 2k. \end{cases} \tag{4.16}$$

Using (4.1) and (4.2), Proposition 4.16 gives,

$$\begin{aligned} &\dim D(\ell,\, \ell\lambda) \dim D(m,\, k\theta) \\ &= \begin{cases} \dim D\big(t_{w_0\lambda}(\ell - m)\Lambda_0,\, t_{w_0(\lambda+\theta)}(m\Lambda_0 + (m-k)\theta)\big), & m \leq 2k, \\ \dim D\big(t_{w_0\lambda}(\ell - m)\Lambda_0,\, t_{w_0\lambda}w_0(m\Lambda_0 + k\theta)\big), & m \geq 2k. \end{cases} \end{aligned} \tag{4.17}$$

Since $\lambda = \lambda_1 + \cdots + \lambda_p$, we have from Theorem 4.2 that

$$\dim D(\ell,\, \ell\lambda) = \prod_{j=1}^{p} \dim D(\ell,\, \ell\lambda_j). \tag{4.18}$$

The proof of (4.16) follows from Proposition 4.12, by using (4.15), (4.17), and (4.18). $\square$

5. The connection with Chari-Venkatesh modules

In this section, we prove that the defining relations of $\mathbf{V}^{\ell,\, \ell\lambda}_{m,\, k\theta}$ can be simplified. This allows us to make connection with the modules introduced by Chari and Venkatesh in [4].

5.1. Let us begin with recalling the notations and definition given in [4]. For $r, s \in \mathbb{Z}_{\geq 0}$, let

$$\mathbf{S}(r,s) = \{(b_p)_{p\geq 0} : b_p \in \mathbb{Z}_{\geq 0},\, \sum_{p\geq 0} b_p = r,\, \sum_{p\geq 0} pb_p = s\}.$$

For $\alpha \in R^+$ and $r, s \in \mathbb{Z}_{\geq 0}$, define an element $\mathbf{x}^-_\alpha(r,s) \in \mathbf{U}(\mathfrak{g}[t])$ by

$$\mathbf{x}^-_\alpha(r,s) = \sum_{(b_p)\in \mathbf{S}(r,s)} (x^-_\alpha \otimes 1)^{(b_0)} (x^-_\alpha \otimes t)^{(b_1)} \cdots (x^-_\alpha \otimes t^s)^{(b_s)}, \tag{5.1}$$

where $X^{(b)}$ denotes the divided power $X^b/b!$. For $K \in \mathbb{Z}_{\geq 0}$, let $\mathbf{S}(r,s)_K$ (resp. ${}_K\mathbf{S}(r,s)$) be the subset of $\mathbf{S}(r,s)$ consisting of elements $(b_p)_{p\geq 0}$, satisfying $b_p = 0$ for $p \geq K$ (resp. $b_p = 0$ for $p < K$). For $\alpha \in R^+$ and $r, s, K \in \mathbb{Z}_{\geq 0}$, define elements $\mathbf{x}^-_\alpha(r,s)_K$ and ${}_K\mathbf{x}^-_\alpha(r,s)$ of $\mathbf{U}(\mathfrak{g}[t])$ by

$$\mathbf{x}^-_\alpha(r,s)_K = \sum_{(b_p)\in \mathbf{S}(r,s)_K} (x^-_\alpha \otimes 1)^{(b_0)} (x^-_\alpha \otimes t^1)^{(b_1)} \cdots (x^-_\alpha \otimes t^{K-1})^{(b_{K-1})}, \tag{5.2}$$

$${}_K\mathbf{x}^-_\alpha(r,s) = \sum_{(b_p)\in {}_K\mathbf{S}(r,s)} (x^-_\alpha \otimes t^K)^{(b_K)} (x^-_\alpha \otimes t^{K+1})^{(b_{K+1})} \cdots (x^-_\alpha \otimes t^s)^{(b_s)}. \tag{5.3}$$

Given $\mu \in P^+$ and a $|R^+|$-tuple $\boldsymbol{\xi} = (\xi(\alpha))_{\alpha\in R^+}$ of partitions such that $|\xi(\alpha)| = \langle \mu,\, \alpha^\vee\rangle$, $\forall\, \alpha \in R^+$. The module $V(\boldsymbol{\xi})$ is the cyclic $\mathfrak{g}[t]$-module generated by $v_{\boldsymbol{\xi}}$ with defining relations:

$$(x_i^+ \otimes t^s)\, v_{\boldsymbol{\xi}} = 0, \quad (\alpha_i^\vee \otimes t^s)\, v_{\boldsymbol{\xi}} = \delta_{s,0}\langle \mu,\, \alpha_i^\vee\rangle\, v_{\boldsymbol{\xi}}, \quad \left(x_i^- \otimes 1\right)^{\langle \mu,\, \alpha_i^\vee\rangle + 1} v_{\boldsymbol{\xi}} = 0, \quad \forall\ s \geq 0, i \in I, \tag{5.4}$$

$$\mathbf{x}_\alpha^-(r,s)\, v_{\boldsymbol{\xi}} = 0, \quad \forall\ \alpha \in R^+, s, r \in \mathbb{N} \text{ such that } s + r \geq 1 + rK + \sum_{j\geq K+1} \xi(\alpha)_j \text{ for some } K \in \mathbb{N}. \tag{5.5}$$

It is proved in [4] that the relations (5.5) may be replaced with the following:

$$_K\mathbf{x}_\alpha^-(r,s)\, v_{\boldsymbol{\xi}} = 0, \quad \forall\ \alpha \in R^+, s, r, K \in \mathbb{N} \text{ such that } s + r \geq 1 + rK + \sum_{j\geq K+1} \xi(\alpha)_j. \tag{5.6}$$

5.2. For $k, \ell, m \in \mathbb{N}$ such that $\ell \geq m \geq k$ and $\lambda \in L^+$, we define three $|R^+|$-tuple of partitions as follows:

$$\boldsymbol{\xi}(\ell,\, \ell\lambda) := (\xi(\ell,\, \ell\lambda)(\alpha))_{\alpha\in R^+}, \quad \text{where} \quad \xi(\ell,\, \ell\lambda)(\alpha) := \Big((d_\alpha \ell)^{(\lambda|\alpha)}\Big),$$

$$\boldsymbol{\xi}(m,\, k\theta) := (\xi(m,\, k\theta)(\alpha))_{\alpha\in R^+}, \quad \text{where} \quad \xi(m,\, k\theta)(\alpha) := \begin{cases} \emptyset, & (\theta|\alpha) = 0,\\ (d_\alpha k), & (\theta|\alpha) = 1,\\ (m,\, 2k - m), & \alpha = \theta \text{ and } m \leq 2k,\\ (2k), & \alpha = \theta \text{ and } m \geq 2k, \end{cases}$$

$$\boldsymbol{\xi}^{\ell,\, \ell\lambda}_{m,\, k\theta} := \Big(\xi^{\ell,\, \ell\lambda}_{m,\, k\theta}(\alpha)\Big)_{\alpha\in R^+}, \quad \text{where} \quad \xi^{\ell,\, \ell\lambda}_{m,\, k\theta}(\alpha) := \begin{cases} \big((d_\alpha\ell)^{(\lambda|\alpha)}\big), & (\theta|\alpha) = 0,\\ \big((d_\alpha\ell)^{(\lambda|\alpha)},\, d_\alpha k\big), & (\theta|\alpha) = 1,\\ \big(\ell^{(\lambda|\theta)},\, m,\, 2k - m\big), & \alpha = \theta \text{ and } m \leq 2k,\\ \big(\ell^{(\lambda|\theta)},\, 2k\big), & \alpha = \theta \text{ and } m \geq 2k. \end{cases} \tag{5.7}$$

The following isomorphisms follow from [4, Theorem 2]:

$$V(\boldsymbol{\xi}(\ell,\, \ell\lambda)) \cong_{\mathfrak{g}[t]} D(\ell,\, \ell\lambda) \qquad \text{and} \qquad V(\boldsymbol{\xi}(m,\, k\theta)) \cong_{\mathfrak{g}[t]} D(m,\, k\theta). \tag{5.8}$$

We are now in position to state the main result of this section.

Theorem 5.1. *Let $k, \ell, m \in \mathbb{N}$ be such that $\ell \geq m \geq k$ and $\lambda \in L^+$. Then there exists an isomorphism of $\mathfrak{g}[t]$-modules,*

$$V(\boldsymbol{\xi}^{\ell,\, \ell\lambda}_{m,\, k\theta}) \cong \mathbf{V}^{\ell,\, \ell\lambda}_{m,\, k\theta}.$$

Remark 5.2. *For $\mathfrak{g} = \mathfrak{sl}_2$, it is proved in [4, §6] that the modules $V(\boldsymbol{\xi})$ are fusion products of evaluation modules $\mathrm{ev}_0\, V(r\varpi_1), r \in \mathbb{Z}_{\geq 0}$ and vice-versa. Using this, Theorem 5.1 gives us for $n \in \mathbb{Z}_{\geq 0}$ that*

$$\mathbf{V}^{\ell,\, \ell n\varpi_1}_{m,\, k\theta} \cong_{\mathfrak{sl}_2[t]} \begin{cases} \big(\mathrm{ev}_0\, V(\ell\varpi_1)\big)^{*n} * \mathrm{ev}_0\, V(m\varpi_1) * \mathrm{ev}_0\, V((2k-m)\varpi_1), & m \leq 2k,\\ \big(\mathrm{ev}_0\, V(\ell\varpi_1)\big)^{*n} * \mathrm{ev}_0\, V(2k\varpi_1), & m \geq 2k. \end{cases}$$

The following corollary shows that certain types of generalized Demazure modules also belong to the family of modules defined in [4].

Corollary 5.3. *Let $k, \ell, m \in \mathbb{N}$ be such that $\ell \geq m \geq k$. Let $\lambda_1, \ldots, \lambda_p$ be a sequence of elements of L^+, and denote $\lambda = \lambda_1 + \cdots + \lambda_p$. Then we have the following isomorphisms of $\mathfrak{g}[t]$-modules:*

$$\begin{aligned}
&V(\boldsymbol{\xi}(\ell,\, \ell\lambda_1)) * \cdots * V(\boldsymbol{\xi}(\ell,\, \ell\lambda_p)) * V(\boldsymbol{\xi}(m,\, k\theta)) \\
&\cong D(\ell,\, \ell\lambda_1) * \cdots * D(\ell,\, \ell\lambda_p) * D(m,\, k\theta) \\
&\cong \mathbf{V}^{\ell,\, \ell\lambda}_{m,\, k\theta} \cong V(\boldsymbol{\xi}^{\ell,\, \ell\lambda}_{m,\, k\theta}) \cong \begin{cases} D\big(t_{w_0\lambda}(\ell-m)\Lambda_0,\, t_{w_0(\lambda+\theta)}(m\Lambda_0 + (m-k)\theta)\big), & m \leq 2k, \\ D\big(t_{w_0\lambda}(\ell-m)\Lambda_0,\, t_{w_0\lambda}w_0(m\Lambda_0 + k\theta)\big), & m \geq 2k. \end{cases}
\end{aligned}$$

Proof. The proof is immediate from Theorem 3.3, (5.8), and Theorem 5.1. $\square$

5.3. **Proof of Theorem 5.1.** The proof is immediate from the following two propositions.

Proposition 5.4. *With hypothesis and notation as in Theorem 5.1, the assignment $v_{\boldsymbol{\xi}^{\ell,\, \ell\lambda}_{m,\, k\theta}} \mapsto v^{\ell,\, \ell\lambda}_{m,\, k\theta}$ gives a surjective morphism from $V(\boldsymbol{\xi}^{\ell,\, \ell\lambda}_{m,\, k\theta})$ onto $\mathbf{V}^{\ell,\, \ell\lambda}_{m,\, k\theta}$.*

Proposition 5.5. *With hypothesis and notation as in Theorem 5.1, the assignment $v^{\ell,\, \ell\lambda}_{m,\, k\theta} \mapsto v_{\boldsymbol{\xi}^{\ell,\, \ell\lambda}_{m,\, k\theta}}$ gives a surjective morphism from $\mathbf{V}^{\ell,\, \ell\lambda}_{m,\, k\theta}$ onto $V(\boldsymbol{\xi}^{\ell,\, \ell\lambda}_{m,\, k\theta})$.*

5.3.1. *Proof of Proposition 5.4.* The proof uses arguments of the proof of [4, Theorem 1]. We need to show that $v = v^{\ell,\, \ell\lambda}_{m,\, k\theta}$ satisfies the defining relations of $V(\boldsymbol{\xi}^{\ell,\, \ell\lambda}_{m,\, k\theta})$. The relations (5.4) with $\mu = \ell\lambda + k\theta$ are clear from (3.1). We now prove the relations (5.5). Let $\alpha \in R^+$ and let s_α denote the number of non-zero parts of $\xi(\alpha) = \xi^{\ell,\, \ell\lambda}_{m,\, k\theta}(\alpha)$. In the cases when (i) $r \geq \xi(\alpha)_1$, (ii) $r \leq \xi(\alpha)_{s_\alpha}$, and (iii) $\xi(\alpha)_{s_\alpha - 1} > r > \xi(\alpha)_{s_\alpha}$, the proof follows as in the proof of [4, Theorem 1] by using the relations (3.2), (3.3) with $i = 0$, and (3.4)-(3.6).

We now prove the relations (5.5) in the remaining case when $\xi(\alpha)_1 > r \geq \xi(\alpha)_{s_\alpha - 1}$. We observe from (5.7) that this case is possible only when $\alpha = \theta$ and $m \leq 2k$. In this case, we have

$$s_\theta = (\lambda|\theta) + 2 \qquad \text{and} \qquad \xi(\theta)_j = \begin{cases} \ell, & 1 \leq j \leq s_\theta - 2, \\ m, & j = s_\theta - 1, \\ 2k - m, & j = s_\theta. \end{cases}$$

We observe that if $(b_p)_{p \geq 0} \in \mathbf{S}(r, s)$ is such that $b_p > 0$ for some $p \geq s_\theta$, then by the relation (3.2) with $\alpha = \theta$, we have

$$\big((x^-_\theta \otimes 1)^{(b_0)} \cdots (x^-_\theta \otimes t^p)^{(b_p)} \cdots (x^-_\theta \otimes t^s)^{(b_s)}\big)\, v = 0.$$

Hence we get

$$\big(\mathbf{x}^-_\theta(r, s) - \mathbf{x}^-_\theta(r, s)_{s_\theta}\big)\, v = 0. \tag{5.9}$$

If $(b_p)_{p \geq 0} \in \mathbf{S}(r, s)_{s_\theta}$, then $s = (s_\theta - 1)b_{s_\theta - 1} + \cdots + b_1$. We consider two cases. First suppose that $b_{s_\theta - 1} > 2k - m$. Then, using the relation (3.4) we have

$$\big((x^-_\theta \otimes 1)^{(b_0)} \cdots (x^-_\theta \otimes t^{s_\theta - 2})^{(b_{s_\theta - 2})} (x^-_\theta \otimes t^{s_\theta - 1})^{(b_{s_\theta - 1})}\big)\, v = 0. \tag{5.10}$$

In the second case, we have $b_{s_\theta - 1} \leq 2k - m$. If we prove that

$$b_{s_\theta - 2} \geq 2(k - b_{s_\theta - 1}) + 1, \tag{5.11}$$

then (5.10) would follow in this case from the relations (3.3) with $\alpha = \theta$ and $i = b_{s_\theta - 1}$. This would give us $\mathbf{x}_\theta^-(r,s)_{s_\theta}\, v = 0$. Using (5.9), we will have $\mathbf{x}_\theta^-(r,s)\, v = 0$ which would complete the proof.

To prove (5.11), we observe that

$$(s_\theta - 1)b_{s_\theta-1} + (s_\theta - 2)b_{s_\theta-2} + (s_\theta - 3)(r - b_{s_\theta-1} - b_{s_\theta-2}) \geq s \geq 1 + r(K-1) + \sum_{j \geq K+1} \xi(\theta)_j,$$

which implies

$$2b_{s_\theta-1} + b_{s_\theta-2} \geq 1 + r(K - s_\theta + 2) + \sum_{j \geq K+1} \xi(\theta)_j.$$

Since $r \geq \xi(\theta)_{s_\theta-1}$, we see that (5.11) is immediate if $K \geq s_\theta - 1$. If $K < s_\theta - 1$, then

$$2b_{s_\theta-1} + b_{s_\theta-2} \geq 1 + \sum_{s_\theta - 1 > j \geq K+1} (\xi(\theta)_j - r) + \xi(\theta)_{s_\theta-1} + \xi(\theta)_{s_\theta} \geq 2k+1,$$

where the last inequality is because $\xi(\theta)_j = \xi(\theta)_1 > r$ for all $1 \leq j \leq s_\theta - 2$. □

5.3.2. *Proof of Proposition 5.5.* The following lemma is crucial in proving the proposition.

Lemma 5.6. *Let $k, \ell, m \in \mathbb{N}$ be such that $\ell \geq m \geq k$ and $\lambda \in L^+$. Then, for all $\alpha \in R^+ \setminus \{\theta\}$ and $1 \leq i \leq k$, the relation*

$$\big(x_\alpha^- \otimes t^{(\lambda|\alpha)}\big)^{\langle (k-i)\theta,\, \alpha^\vee\rangle + 1} \big(x_\theta^- \otimes t^{(\lambda|\theta)+1}\big)^i\, v_{m,\,k\theta}^{\ell,\,\ell\lambda} = 0, \tag{5.12}$$

is redundant in the definition of $\mathbf{V}_{m,\,k\theta}^{\ell,\,\ell\lambda}$.

Proof. To prove the lemma, we show by considering two cases that the relation (5.12) follows from the relations (3.2) and (3.3) with $i = 0$.

Case (1). *Suppose that α is either a long root or a short simple root.*

Since $\alpha \in R^+ \setminus \{\theta\}$, it is easy to see that $(\theta|\alpha)$ is either 0 or 1. If $(\theta|\alpha) = 0$, then (5.12) follows from (3.2). Assume that $(\theta|\alpha) = 1$. We may also assume that $d_\alpha \in \{1, 2\}$. Indeed, if $d_\alpha = 3$, then $\mathfrak{g}$ is of type G_2 and α is the short simple root of $\mathfrak{g}$, hence $(\theta|\alpha) = 0$. We now have

$$(\theta | r_\alpha(\theta)) = (\theta | \theta - d_\alpha \alpha) = 2 - d_\alpha \in \{0, 1\} \qquad \text{and} \qquad (\theta - \alpha), r_\alpha(\theta) \in R^+. \tag{5.13}$$

Let $A_\alpha, B_\alpha \in \mathbb{C} \setminus \{0\}$ be such that $[x_{r_\alpha(\theta)}^-, x_\alpha^-] = A_\alpha x_{\theta - (d_\alpha - 1)\alpha}^-$ and $[x_{\theta-\alpha}^-, x_\alpha^-] = B_\alpha x_\theta^-$. We denote

$$f(A_\alpha, B_\alpha) = \begin{cases} A_\alpha = B_\alpha, & d_\alpha = 1, \\ \frac{A_\alpha B_\alpha}{2}, & d_\alpha = 2. \end{cases}$$

Set $X = x_{r_\alpha(\theta)}^- \otimes t^{(\lambda|r_\alpha(\theta))+1}, Y = x_\alpha^- \otimes t^{(\lambda|\alpha)}, Z = x_\theta^- \otimes t^{(\lambda|\theta)+1},$ and $v = v_{m,\,k\theta}^{\ell,\,\ell\lambda}$. We observe that

$$[X, Z] = [Y, Z] = [[X, Y], Z] = 0, \quad d_\alpha f(A_\alpha, B_\alpha) Z = \begin{cases} [X, Y], & d_\alpha = 1, \\ [[X, Y], Y], & d_\alpha = 2, \end{cases} \qquad \text{and} \quad X\, v = 0, \tag{5.14}$$

by (3.2) and (5.13).

By acting both sides to the relation $Y^{kd_\alpha+1}\,v = 0$, which holds in this case in $\mathbf{V}^{\ell,\,\ell\lambda}_{m,\,k\theta}$ by the relation (3.3) with $i=0$, with X^i, we get

$$X^i\,Y^{kd_\alpha+1}\,v = 0. \tag{5.15}$$

We claim that for $i' \in \mathbb{Z}_{\geq 0}$,

$$X^{(i')}\,Y^{(k'd_\alpha+1)}\,v = (f(A_\alpha,B_\alpha))^{i'}Y^{((k'-i')d_\alpha+1)}\,Z^{(i')}\,v, \qquad \forall\ k' \geq i', \tag{5.16}$$

where $\mathbb{X}^{(p)}$ denotes the divided power $\mathbb{X}^p/p!$. We observe from (5.15) that the proof of (5.12) in this case follows once we establish the claim.

We now prove the claim (5.16) by induction on i'. For $i'=0$, there is nothing to prove. Now suppose that $i' \geq 1$. Let $k' \geq i'$. By the induction hypothesis, we have

$$X^{(i'-1)}\,Y^{(k'd_\alpha+1)}\,v = (f(A_\alpha,B_\alpha))^{i'-1}Y^{((k'-i'+1)d_\alpha+1)}\,Z^{(i'-1)}\,v. \tag{5.17}$$

By acting both sides to (5.17) with $\frac{1}{i'}X$, we get

$$X^{(i')}\,Y^{(k'd_\alpha+1)}\,v = \frac{(f(A_\alpha,B_\alpha))^{i'-1}}{i'}X\,Y^{((k'-i'+1)d_\alpha+1)}\,Z^{(i'-1)}\,v. \tag{5.18}$$

Since $X\,Z^{(i'-1)}\,v = Z^{(i'-1)}\,X\,v = 0$ by (5.14), we may replace the right hand side of (5.18) by

$$\begin{aligned}&\frac{(f(A_\alpha,B_\alpha))^{i'-1}}{i'}[X,Y^{((k'-i'+1)d_\alpha+1)}]\,Z^{(i'-1)}\,v\\ &= \frac{(f(A_\alpha,B_\alpha))^{i'-1}}{((k'-i'+1)d_\alpha+1)!i'}\sum_{p=0}^{(k'-i'+1)d_\alpha} Y^p\,[X,Y]\,Y^{(k'-i'+1)d_\alpha-p}\,Z^{(i'-1)}\,v.\end{aligned} \tag{5.19}$$

Suppose that $d_\alpha = 1$. Then, since $[X,Y] = f(A_\alpha,B_\alpha)Z$ and $[Z,Y]=0$ by (5.14), it is easily checked that (5.19) proves the claim (5.16). Suppose that $d_\alpha = 2$. Then, since $[[X,Y],Z]=0$ and $[X,Y]\,v=0$ by (3.2), we have

$$\begin{aligned}[X,Y]\,Y^{(k'-i'+1)d_\alpha-p}\,Z^{(i'-1)}\,v &= [[X,Y],Y^{(k'-i'+1)d_\alpha-p}]\,Z^{(i'-1)}\,v\\ &= \sum_{q=0}^{(k'-i'+1)d_\alpha-p-1} Y^q\,[[X,Y],Y]\,Y^{(k'-i'+1)d_\alpha-p-q-1}\,Z^{(i'-1)}\,v.\end{aligned} \tag{5.20}$$

Since $[[X,Y],Y] = 2f(A_\alpha,B_\alpha)Z$ and $[Z,Y]=0$ by (5.14), the right hand side of (5.20) simplifies to

$$2i'f(A_\alpha,B_\alpha)((k'-i'+1)d_\alpha-p)Y^{(k'-i'+1)d_\alpha-p-1}\,Z^{(i')}\,v.$$

Substituting this into (5.19), we get that the right hand side of (5.18) is equal to

$$\begin{aligned}&\frac{2(f(A_\alpha,B_\alpha))^{i'}}{((k'-i'+1)d_\alpha+1)!}\sum_{p=0}^{(k'-i'+1)d_\alpha}((k'-i'+1)d_\alpha-p)Y^{(k'-i'+1)d_\alpha-1}\,Z^{(i')}\,v\\ &= \frac{2(f(A_\alpha,B_\alpha))^{i'}}{((k'-i'+1)d_\alpha+1)!}\frac{((k'-i'+1)d_\alpha)((k'-i'+1)d_\alpha+1)}{2}Y^{(k'-i'+1)d_\alpha-1}\,Z^{(i')}\,v.\end{aligned} \tag{5.21}$$

This proves the claim (5.16).

Case (2). *Suppose that α is a short root.*

We proceed by induction on $\operatorname{ht}\alpha$. If $\operatorname{ht}\alpha = 1$, then (5.12) follows from case (1). Assume that $\operatorname{ht}\alpha > 1$. Then, since α is short, there exist a short root $\beta \in R^+$ and a root $\gamma \in R^+$ such that $\alpha = \beta + \gamma$. Set $v_i = \left(x_\theta^- \otimes t^{(\lambda|\theta)+1}\right)^i v_{m,\,k\theta}^{\ell,\,\ell\lambda}$. By induction hypothesis, we have

$$\left(x_\beta^- \otimes t^{(\lambda|\beta)}\right)^{\langle (k-i)\theta,\,\beta^\vee\rangle+1} v_i = 0. \tag{5.22}$$

If γ is long (resp. short), then from case (1) (resp. induction hypothesis), we have

$$\left(x_\gamma^- \otimes t^{(\lambda|\gamma)}\right)^{\langle (k-i)\theta,\,\gamma^\vee\rangle+1} v_i = 0. \tag{5.23}$$

It is easily checked that the Lie subalgebra of $\mathfrak{g}[t]$ generated by $\{x_\beta^- \otimes t^{(\lambda|\beta)}, x_\gamma^- \otimes t^{(\lambda|\gamma)}\}$ is isomorphic to the nilradical of the Borel subalgebra of the Lie algebra of type A_2 (resp. B_2, G_2), if γ is short (resp. γ is long and $d_\beta = 2$, γ is long and $d_\beta = 3$). Since

$$\alpha^\vee = \begin{cases} \beta^\vee + d_\alpha \gamma^\vee, & \gamma \text{ is long},\\ \beta^\vee + \gamma^\vee, & \gamma \text{ is short},\end{cases}$$

we have from (5.22) and (5.23) (see [13, Lemma 4.5]) that

$$\left(x_\alpha^- \otimes t^{(\lambda|\alpha)}\right)^{\langle (k-i)\theta,\,\beta^\vee\rangle + d_\alpha\langle (k-i)\theta,\,\gamma^\vee\rangle+1} v_i = 0, \qquad \text{if } \gamma \text{ is long},$$

and

$$\left(x_\alpha^- \otimes t^{(\lambda|\alpha)}\right)^{\langle (k-i)\theta,\,\beta^\vee\rangle + \langle (k-i)\theta,\,\gamma^\vee\rangle+1} v_i = 0, \qquad \text{if } \gamma \text{ is short},$$

which are equivalent to (5.12). Hence the lemma. □

We now prove Proposition 5.5. Let $\boldsymbol{\xi} = \boldsymbol{\xi}_{m,\,k\theta}^{\ell,\,\ell\lambda}$. We need to show that $v_{\boldsymbol{\xi}}$ satisfies the defining relations of $\mathbf{V}_{m,\,k\theta}^{\ell,\,\ell\lambda}$. The relations (3.1) are clear from (5.4). By taking $r = 1, s = (\lambda+\theta|\alpha)$, and $K = (\lambda+\theta|\alpha)$ in (5.5), we obtain

$$\left(x_\alpha^- \otimes t^{(\lambda+\theta|\alpha)}\right) v_{\boldsymbol{\xi}} = 0, \quad \forall\ \alpha \in R^+, \tag{5.24}$$

which proves the relations (3.2).

To prove the relations (3.3), using Lemma 5.6, it is enough to prove that the following relations hold in $V(\boldsymbol{\xi})$

$$\left(x_\alpha^- \otimes t^{(\lambda|\alpha)}\right)^{\langle k\theta,\,\alpha^\vee\rangle+1} v_{\boldsymbol{\xi}} = 0, \quad \forall\ \alpha \in R^+, \tag{5.25}$$

$$\left(x_\theta^- \otimes t^{(\lambda|\theta)}\right)^{2(k-i)+1}\left(x_\theta^- \otimes t^{(\lambda|\theta)+1}\right)^i v_{\boldsymbol{\xi}} = 0, \quad \forall\ 0 \le i \le k. \tag{5.26}$$

Using (5.24), the relations (5.25) follow from (5.6), by taking $r = \langle k\theta,\,\alpha^\vee\rangle + 1$, $s = r(\lambda|\alpha)$, and $K = (\lambda|\alpha)$. Using the relation $\left(x_\theta^- \otimes t^{(\lambda|\theta)+2}\right) v_{\boldsymbol{\xi}} = 0$, the relations (5.26) (resp. (3.4), (3.5)) follow from (5.6) with $\alpha = \theta$, by taking $r = 2k - i + 1$ (resp. $r = 2k - m + 1$, $r = 1$), $s = r(\lambda|\theta) + i$ (resp. $s = r(\lambda|\theta) + r$, $s = (\lambda|\theta) + 1$), and $K = (\lambda|\theta)$ (resp. $K = (\lambda|\theta) + 1$, $K = (\lambda|\theta) + 1$). This completes the proof of the proposition. □

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